BIOS 760 Midterm Exam I (Fall, 2016)

1. Assume $X \sim MN(0, \Sigma)$ where Σ is a $d \times d$ positive definite matrix. Let α and β be two $d \times 1$ constant vectors and both are non-zeros. We aim to derive the distribution of $(\alpha' X)/|\beta' X|$.

   (a) (2 points) Find what is the joint distribution of $(\alpha' X, \beta' X)$.

   (b) (3 points) Show that $\alpha' X = a(\beta' X) + Z$, where Z is independent of $\beta' X$ and a is a scale. Determine a and Z's distribution. Thus,

   $$\frac{\alpha' X}{|\beta' X|} = a + \frac{Z}{|\beta' X|}.$$

   (c) (5 points) Give the exact distribution of $\alpha' X / |\beta' X|$.

2. Let $\Omega_1 = \{1, 2, ...\}$ and $\Omega_2 = R$. Consider the product measure space

   $$\left\{\Omega_1 \times \Omega_2, \mu \times \lambda_F, 2^{\Omega_1} \times \mathcal{B}\right\},$$

   where μ is a probability measure with $\mu(\{k\}) = 2^{-k}$, $\mathcal{B}$ is the Borel-σ field, and λ_F is the LS-measure generated by distribution function $F(x) = \{1 - e^{-x}\} I(x > 0)$.

   (a) (1 point) Show that this product measure space is a probability measure space.

   (b) (3 points) Define a function on $\Omega_1 \times \Omega_2$ as $X(w_1, w_2) = I(w_2 \in [w_1, w_1 + 1))$. Show that X is a measurable function so it is a random variable from (a).

   (c) (3 points) Use Fubin's theorem to calculate $E[X] = \int_{\Omega_1 \times \Omega_2} X(\omega_1, \omega_2) d(\mu \times \lambda_F)$..

   (d) (2 points) Define another two functions $Y(w_1, w_2) = w_1$ and $Z(w_1, w_2) = w_2$. Show that Y and Z are two independent random variables.

   (e) (3 points) What are the density functions of Y and Z? Specific the dominating measures in the real space so that the densities exist.

   (f) (3 points) Clearly, $X = I(Z \in [Y, Y + 1))$. Use the joint distribution of (Y, Z) to calculate $E[X]$.