BIOS 760 MIDTERM I, 2014: Solution

1. (a) $E[X|\sigma(Y,Z)] =$

$$\Sigma_{x,yz}\Sigma_{yz}^{-1}\begin{pmatrix} Y \\ Z \end{pmatrix} = \frac{\begin{pmatrix} a, & 0 \end{pmatrix}\begin{pmatrix} 1 & -b \\ -b & 1 \end{pmatrix}\begin{pmatrix} Y \\ Z \end{pmatrix}}{1-b^2} = \frac{aY - abZ}{1-b^2}.$$

(b) $Y|\sigma(Z) \sim N(bZ, 1-b^2)$ which implies $E[Y^2|\sigma(Z)] = b^2Z^2 + 1 - b^2$.

(c) $E[XYZ^2] = E\left[YZ^2 E[X|\sigma(Y,Z)]\right]$

$$\begin{aligned} &= E\left[\frac{YZ^2(aY - abZ)}{1-b^2}\right] \\ &= \frac{aE[Y^2Z^2] - abE[YZ^3]}{1-b^2} \\ &= \frac{aE\left[Z^2E[Y^2|Z]\right] - abE\left[Z^3E[Y|Z]\right]}{1-b^2} \\ &= \frac{aE[b^2Z^4 + (1-b^2)Z^2] - ab^2EZ^4}{1-b^2} \\ &= a. \end{aligned}$$

2. Suppose $\exists A \in \mathcal{A} : \mu(A) = 0$ but $\int_A X d\mu > 0$. Then $\exists$ a sequence of simple functions $0 \le X_n \uparrow X$ such that $\int_A X_n d\mu \to \int_A Z d\mu$. But this implies, for some $\delta > 0$ and integer $n < \infty$,

$$\delta \le \int_A X_n d\mu = \sum_{j=1}^{k_n} x_{jn}\mu(A \cap B_{jn}) = 0,$$

which is a contradiction! Thus the desired result holds.

3. (a) Showing ν is measurable follows from its easy-to-verify additivity on finite disjoint sets combined with the Caratheodory Extension Theorem. Now we will show that it is also σ-finite. Since μ is σ-finite, $\exists$ a countable collection $A_1, A_2, \ldots \in \mathcal{A}$ such that $\bigcup_{j\ge1} A_j = \Omega$ and $\mu(A_j) < \infty$ for all j. Now consider the countable collection of sets $\{A_j \cap B_k : j,k \ge 1\}$, and note that the union of these sets is Ω. Moreover, $\int_{A_j \cap B_k} Y d\mu \le k\mu(A_j) < \infty$. Similar arguments verify that λ is also a σ-finite measure.

(b) Suppose $\mu(A) = 0$. Then $\nu(A) = 0$ by the result of Problem 2. Also, if $\nu(A) = 0$, then $\lambda(A) = \int_A X d\nu = 0$ by reapplication of the Problem 2 result.

(c) Existence follows from μ, ν and λ all being σ-finite measures (although we only need σ-finiteness for μ and ν at this point but will need it for λ later). The forms are $\frac{d\nu}{d\mu} = Y$, $\frac{d\lambda}{d\mu} = XY$, and $\frac{d\lambda}{d\nu} = X$.

(d) i. Suppose $\lambda(A) = 0$. Then

$$\begin{aligned}\nu(A) &= \int_A Y d\mu \\ &= \int_{A\cap C} Y d\mu = \int_{A\cap B\cap C} Y d\mu + \int_{A\cap B^c\cap C} Y d\mu \\ &= \int_{A\cap B\cap C} Y d\mu,\end{aligned}$$

since $\int_{A\cap B^c\cap C} Y d\mu = 0$ from the fact that $\mu(B^c \cap C) = 0$ and the result of Problem 2. However,

$$\int_{A\cap B\cap C} Y d\mu = \int_{A\cap B} Y d\mu = \int_A \left[X^{-1}1\{X > 0\}\right] XY d\mu = \int_A U d\lambda,$$

where $U = X^{-1}1\{X > 0\}$ is measurable. Now $\int_A U d\lambda = 0$ by reapplication of the Problem 2 result. Thus $\nu \prec\prec \lambda$.

ii. Since λ is σ-finite by part 3.(a), we have that $\frac{d\nu}{d\lambda}$ exists and equals $X^{-1}1\{X > 0\}$.