BIOS 760 MIDTERM I, 2014

1. Let $(X, Y, Z)^T$ be multivariate normal with mean $(0, 0, 0)^T$ and covariance

$$\Sigma = \begin{pmatrix} 1 & a & 0 \\ a & 1 & b \\ 0 & b & 1 \end{pmatrix},$$

where $a^2 + b^2 < 1$. Do the following:

   (a) (3 points) Show that $E[X|\sigma(Y, Z)] = \frac{aY - abZ}{1-b^2}$.

   (b) (3 points) Show that $E[Y^2|\sigma(Z)] = b^2 Z^2 + 1 - b^2$.

   (c) (4 bonus points) Derive $E[XYZ^2]$.

2. (4 points) Let $X \geq 0$ be a measurable function on a measure space $(\Omega, \mathcal{A}, \mu)$. Show that for every $A \in \mathcal{A}$, if $\mu(A) = 0$, then $\int_A X d\mu = 0$.

3. Let $X, Y \geq 0$ be measurable functions on a σ-finite measure space $(\Omega, \mathcal{A}, \mu)$. For any $A \in \mathcal{A}$, define the set functions $\nu(A) = \int_A Y d\mu$ and $\lambda(A) = \int_A XY d\mu$. Do the following:

   (a) (5 points) Show that ν and λ are σ-finite measures. (Hint: Consider the countable collection of sets $\{B_k, k \geq 1\}$, where $B_k = \{\omega : k - 1 \leq Y(\omega) < k\}$ for all $k \geq 1$.)

   (b) (3 points) Show that $\lambda \prec\prec \nu \prec\prec \mu$.

   (c) (2 points) Verify existence and derive the form of the Radon-Nikodym derivatives $\frac{d\nu}{d\mu}$, $\frac{d\lambda}{d\mu}$ and $\frac{d\lambda}{d\nu}$.

   (d) Let $B = \{\omega : X(\omega) > 0\}$, $C = \{\omega : Y(\omega) > 0\}$ and assume $\mu(B^c \cap C) = 0$. Do the following:

      i. (4 bonus points) Show that $\nu \prec\prec \lambda$.

      ii. (2 bonus points) Verify existence and derive the form of the Radon-Nikodym derivative $\frac{d\nu}{d\lambda}$.