BIOS 760 MIDTERM I, 2012: Solution

1. (a) Note that
$$p_\theta(x) = \exp\left[-(1/2\theta)x^2 - \log(\theta)\right] x.$$
Thus if we let $\eta(\theta) = -1/(2\theta)$, the density obtains the canonical form:
$$\exp[\eta(\theta)T(x) - A(\eta)]h(x),$$
where $T(x) = x^2$ and $A(\eta) = \log[-1/(2\eta)]$ and $h(x) = x$.

   (b) Because this is an exponential family in canonical form, the MGF has the form
$$\exp[A(\eta+t) - A(\eta)] = \frac{\eta}{\eta+t} = (1-2\theta t)^{-1}.$$

2. (a) Let $B \subset \Omega$ be the set of ω such that $|X(\omega)| < \infty$, and note by the stated conditions, $\mu(B^c) = 0$. Thus $Y_n \to_{a.e.} 0$. Since $\mu(\Omega) < \infty$, this almost everywhere convergence implies $Y_n \to_\mu 0$.

   (b) Note that for each $\omega \in \Omega$, $Y_n(\omega) \to 0$. Since $|Y_n| \leq Z$, for the constant function $Z = 1$, and $\mu(\Omega) < \infty$, we can invoke the DCT to bring the limit through the integral, and the desired result follows.

3. (a) Since Y_2 is independent of X_1, we obtain $E\left[X_1 | \aleph_1\right] = E\left[X_1 | Y_1\right]$, and the rest follows from the form of the conditional distribution of dependent normals.

   (b) Note that $\aleph_1$ is the smallest σ-field generated by sets of the form $\{Y_1 \leq u_1\} \cap \{Y_2 \leq u_2\}$ for any $u_1, u_2 \in R$. Since $\aleph_2$ is the smallest σ-field generated by sets of the form $\{Y_1 \leq u_1\} \cap \{Y_2 \leq u_1\}$, we can easily obtain the generating sets for $\aleph_2$ from the generating sets for $\aleph_1$ by setting $u_2 = u_1$. Thus $\aleph_2 \subset \aleph_1$.

   (c) Since $\aleph_2 \subset \aleph_1$,
$$E[X_1|\aleph_2] = E\left[E[X_1|\aleph_1] | \aleph_2\right] = 2\rho E[Y_1|Z].$$
Allowing $g(z) = E[Y_1|Z = z]$, we obtain by the definition of conditional expectation that
$$\int_{Z \leq u} g(Z) dP = \int_{Z \leq u} Y_1 dP. \qquad (1)$$

The CDF of the minimum of two independent standard normals is $\Phi^2(u)$, with density $2\phi(u)\Phi(u)$. Thus the left-hand-side of (1) is $\int_{-\infty}^{u} g(u)2\phi(u)\Phi(u)du$ with derivative $g(u)2\phi(u)\Phi(u)$. On the other hand, the right-hand-side of (1) equals

$$\begin{aligned} E[Y_1 1(Y_1 \leq u)]E[1(Y_2 \leq u)] &= \left[\int_{-\infty}^{u} z\phi(z)dz\right][\Phi(u)] \\ &= -\phi(u)\Phi(u), \end{aligned}$$

and thus the derivative of the right-hand-side of (1) is $u\phi(u)\Phi(u) - \phi^2(u)$. By setting the two derivatives equal, we obtain that

$$g(z) = \frac{z}{2} - \frac{\phi(z)}{2\Phi(z)}.$$

The desired results follows now by combining this with part (a) above.