BIOS 760 MIDTERM I, 2012

1. Let X be a positive random variable with density:

$$p_\theta(x) = \frac{x}{\theta} e^{-x^2/(2\theta)}, \quad x > 0,$$

where $0 < \theta < \infty$. Do the following:

   (a) (3 points) Show that $\{p_\theta\}$ is a one-parameter exponential family with canonical parameter $\eta(\theta) = -1/(2\theta)$.

   (b) (3 points) Show that the moment generating function for $T(X) = X^2$ is $M_{T(X)}(t) = (1 - 2\theta t)^{-1}$.

2. Let $X : \Omega \mapsto R$ be a measurable function on the measure space $(\Omega, \mathcal{A}, \mu)$, where $\mu(\Omega) < \infty$ and $|X| < \infty$ almost everywhere, and define

$$Y_n = \left(\frac{X}{1 + |X|} \right)^n .$$

Do the following:

   (a) (3 points) Show that $Y_n \to_{a.e.} 0$ and $Y_n \to_\mu 0$.

   (b) (4 points) Show that $\int Y_n d\mu \to 0$.

3. Let (X_1, Y_1) and (X_2, Y_2) be two independent pairs of random variables with (X_j, Y_j) being bivariate normal with mean zero and covariance

$$\Sigma = \begin{pmatrix} 4 & 2\rho \\ 2\rho & 1 \end{pmatrix},$$

for $j = 1, 2$, where $|\rho| \le 1$. Let $\aleph_1$ be the σ-field generated by Y_1 and Y_2 (i.e., $\aleph_1 = \sigma(Y_1, Y_2)$), and let $\aleph_2$ be the σ-field generated by $Z = \max(Y_1, Y_2)$ (i.e., $\aleph_2 = \sigma(Z)$). Do the following:

   (a) (3 points) Show that $E[X_1 | \aleph_1] = E[X_1 | Y_1] = 2\rho Y_1$.

   (b) (4 points) Show that $\aleph_2 \subset \aleph_1$.

   (c) (5 bonus points) Show that

$$E\left[X_1 | \aleph_2\right] = \rho \left(Z - \frac{\phi(Z)}{\Phi(Z)} \right),$$

   where $\phi(x)$ is the standard normal density at x and $\Phi(x) = \int_{-\infty}^{x} \phi(u) du$.