SOLUTION TO BIOS 760 MIDTERM I, 2011

1. (a) After setting the derivative to zero, we obtain $\hat{\beta}_w = \sum_{i=1}^n w_i Y_i X_i / \sum_{i=1}^n w_i X_i^2$.

   (b) Since $E[\hat{\beta}_w | X_1, ..., X_n] = \beta$, $Var(\hat{\beta}_W) = E[Var(\hat{\beta}_w | X_1, ..., X_n)]$. It suffices to show that $w_1 = ... = w_n = 1$ minimizes $Var(\hat{\beta}_w | X_1, ..., X_n) = \sigma^2 \sum_{i=1}^n w_i^2 X_i^2 / (\sum_{i=1}^n w_i X_i^2)^2$. Note that this term is invariant if w's are scaled by a positive factor. Thus, we assume $\sum_{i=1}^n w_i X_i^2 = 1$ and aim to minimize $\sum_{i=1}^n w_i^2 X_i^2$. Using the Lagrange multiplier, we obtain that $w_1 = ... = w_n$. Thus, the conclusion holds.

   (c) First,
$$\hat{\beta}_w = \beta + \sum_{i=1}^n \epsilon_i X_i / \sum_{i=1}^n X_i^2 = \beta + \frac{\sum_{i=1}^n \epsilon_i X_i}{\sqrt{\sum_{i=1}^n X_i^2}} \frac{1}{\sqrt{\sum_{i=1}^n X_i^2}}.$$
   Since the conditional distribution of $\frac{\sum_{i=1}^n \epsilon_i X_i}{\sqrt{\sum_{i=1}^n X_i^2}}$ given $X_1, ..., X_n$ is $\sigma N(0,1)$, which is independent of $X_1, ..., X_n$, we conclude $\frac{\sum_{i=1}^n \epsilon_i X_i}{\sqrt{\sum_{i=1}^n X_i^2}}$ follows $\sigma N(0,1)$ and is independent of $X_1, ..., X_n$. On the other hand, $X_1^1 + ... + X_n^2 \sim \chi_n^2$. Thus
$$\hat{\beta}_w \sim \beta + \frac{\sigma}{\sqrt{n}} t_{(n)}.$$

2. (a) Clearly, $\nu(\phi) = 0$. If $A_1, A_2, ...$ are disjoint sets in $\mathcal{A}$, then
$$\nu(\cup_i A_i) = \int X(\omega) \sum_{i=1}^\infty I_{A_i}(\omega) dP(\omega) = \int \lim_n \sum_{i=1}^n X(\omega) I_{A_i}(\omega) dP(\omega)$$
$$= \lim_n \int \sum_{i=1}^n X(\omega) I_{A_i}(\omega) dP(\omega) = \sum_{i=1}^\infty \int X(\omega) I_{A_i}(\omega) dP(\omega) = \sum_i \nu(A_i),$$
   where we apply the MCT. Thus, ν is a measure.

   (b) Clearly, $P(A) = 0$ implies $\nu(A) = 0$. The definition of ν gives $d\nu/dP(w) = X(w)$.

   (c) If $P_X(B) = 0$ for a Borel set B, then $P(X^{-1}(B)) = 0$ so $\nu(X^{-1}(B)) = 0$. The last gives $\nu_X(B) = 0$. Thus, $\nu_X \prec\prec P_X$. Furthermore,
$$\nu_X(B) = \int_{X^{-1}(B)} X(\omega) dP(\omega) = \int X(\omega) I(X(\omega) \in B) dP(\omega)$$
$$= \int x I(x \in B) dP_X(x) = \int_B x dP_X(x).$$
   Hence, $d\nu_X / dP_X(x) = x$.