BIOS 760 MIDTERM I, 2011

1. Let $X_1, ..., X_n$ be i.i.d from $N(0,1)$. Suppose that $Y_i = \beta X_i + \epsilon_i, i = 1, ..., n$, where $\epsilon_1, ..., \epsilon_n$ are i.i.d from $N(0, \sigma^2)$ and they are independent of X's. We define a weighted least-square estimator for β as the one minimizing

$$\sum_{i=1}^{n} w_i (Y_i - \beta X_i)^2,$$

where $w = (w_1, ..., w_n)$ is constant and $w_i \geq 0, i = 1, ..., n$,

   (a) (4 points) Show that the weighted least estimator associated with weight w is given as

$$\hat{\beta}_w = \frac{\sum_{i=1}^{n} w_i Y_i X_i}{\sum_{i=1}^{n} w_i X_i^2}.$$

   (b) (4 points) Show that $Var(\hat{\beta}_w)$ is the smallest if we choose $w_1 = ... = w_n = 1$. Hint: first show $w_1 = ... = w_n = 1$ minimizes $Var(\hat{\beta}_w | X_1, ..., X_n)$.

   (c) (4 points) For this optimal weight, what is the distribution of $\hat{\beta}_w$?

2. Let $(\Omega, \mathcal{A}, P)$ be a probability measure space and X is a non-negative measurable function (random variable) defined on Ω. Assume that $E[X] = \int_\Omega X(\omega) dP(\omega)$ is finite. For any set $A \in \mathcal{A}$, we define a set function $\nu(A) = \int_A X(\omega) dP(\omega)$.

   (a) (6 points) Show that ν is a measure in $(\Omega, \mathcal{A})$.

   (b) (6 points) Show that ν is dominated by P. What is the Random-Nikodym derivative $d\nu/dP$?

   (c) (6 points) Note that X is a measurable function from $(\Omega, \mathcal{A}, P)$ to $(R, \mathcal{B})$ where $\mathcal{B}$ is the Borel σ-field. Thus, there exists an X-induced measure, P_X, for $(R, \mathcal{B})$. On the other hand, X is also a measurable function from $(\Omega, \mathcal{A}, \nu)$ to $(R, \mathcal{B})$ so there is another X-induced measure, ν_X, for $(R, \mathcal{B})$. Show that ν_X is dominated by P_X and give the Random-Nikodym derivative $d\nu_X/dP_X$.