BIOS 760: Solution to Midterm I 2010

1. Let $X = (X_1, X_2)$ be a bivariate random variable with density function

$$f(x_1, x_2) = 2\,, \qquad\qquad 0 < x_1 < x_2 < 1\,.$$

   (a) Note that $F_X(a, b) = 0$ if either $a < 0$ or $b < 0$.

   Assume that $0 < a, b < 1$, then

$$\begin{aligned} F_X(a,b) &= \int_0^b \int_0^a f(x_1, x_2) dx_1 dx_2 = \int_0^b \int_0^{\min(a,x_2)} 2 dx_1 dx_2 \\ &= \int_0^b 2\min(a, x_2) dx_2 = \int_0^b (2x_2 - 2(x_2 - a)_+) dx_2 \\ &= b^2 - (b-a)_+^2 \end{aligned}$$

   For $a \geq 1$ we have $F_X(a, b) = F_X(1, b)$ and similarly for $b \geq 1$ we have $F_X(a, b) = F_X(a, 1)$

   (b) For $0 < x_1 < 1$,

$$f(x_1) = \int f(x_1, x_2) dx_2 = \int_{x_1}^1 2 dx_2 = 2(1 - x_1)\,.$$

   For $0 < x_2 < 1$,

$$f(x_2) = \int f(x_1, x_2) dx_1 = \int_0^{x_2} 2 dx_1 = 2x_2\,.$$

   (c) No, since $f(x_1, x_2) \neq f(x_1)f(x_2)$ we conclude that X_1 and X_2 are dependent.

   (d) For $0 < x_1 < x_2$,

$$f(x_1|x_2) = \frac{f(x_1, x_2)}{f(x_2)} = \frac{2}{2x_2} = \frac{1}{x_2}\,.$$

   Hence

$$E[X_1|X_2] = \int_0^{x_2} \frac{x_1}{x_2} dx_1 = \frac{x_2}{2}$$

2. Let $X = (X_1, X_2)$ be as in Question 1.

   Define the function $u(x_1, x_2) = \left(\frac{x_1}{x_2}, x_2\right)$ and denote $Y = (Y_1, Y_2) = u(X_1, X_2)$.

   (a) The Jacobian matrix is the matrix of first-order partial derivatives.

$$J_u = \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} = \begin{pmatrix} 1/x_2 & -1/(x_2)^2 \\ 0 & 1 \end{pmatrix}.$$

   The Jacobian is the determinant of the Jacobian matrix and it equals $1/x_2$.

(b) The density of Y can be computed from the density of X as follows:

$$f_Y(y_1, y2) = f_X(u^{-1}(y_1, y_2))J_u^{-1}.$$

Note that $x_2 = y_2$ and $x_1 = y_1 y_2$ and $J_u^{-1} = x_2 = y_2$. Hence,

$$f_Y(y_1, y2) = 2y_2 1_{\{0<y_1 y_2<y_2<1\}}.$$

Since $0 < y_1 y_2 < y_2 \Leftrightarrow 0 < y_1 < 1$, we can write

$$f_Y(y_1, y2) = 2y_2 1_{\{0<y_1,y_2<1\}}.$$

(c) Yes, Y_1 is independent of Y_2. To see that note that for $0 < y_1, y_2 < 1$,

$$\begin{aligned} f(y_1) &= \int_0^1 f(y_1, y_2) dy_2 = \int_0^1 2y_2 dy_2 = 1 \\ f(y_2) &= \int_0^1 f(y_1, y_2) dy_1 = \int_0^1 2y_2 dy_1 = 2y_2 \end{aligned}$$

Since $f_Y(y_1, y_2) = f(y_1)f(y_2)$ we conclude that Y_1 is independent of Y_2.

3. Let X_1, X_2, X_3 be random variables on some probability space.

(a) Note that since $E[E(X|Y)] = E[X]$, we have

$$E\left[X_1 E(X_2|X_3)\right] = E\left[E\left[X_1 E(X_2|X_3)\right]|X_3\right].$$

Since $E(X_2|X_3)$ is measurable with respect to the σ-field generated by X_3,

$$E\left[E\left[X_1 E(X_2|X_3)\right]|X_3\right] = E\left[E(X_2|X_3)E(X_1|X_3)\right].$$

Similarly, $E[X_2 E(X_1|X_3)] = E[E(X_2|X_3)E(X_1|X_3)]$, and we obtain the required equality.

(b) Let

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \Sigma_{13} \\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23} \\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33} \end{pmatrix}\right).$$

Note that

$$\begin{aligned} E[E(X_1|X_2)|X_3] &= E[\Sigma_{12}\Sigma_{22}^{-1}X_2|X_3] = \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{23}\Sigma_{33}^{-1}X_3 \\ E[E(X_2|X_1)|X_3] &= E[\Sigma_{21}\Sigma_{11}^{-1}X_1|X_3] = \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{13}\Sigma_{33}^{-1}X_3. \end{aligned}$$

Thus when $\Sigma_{22}^{-1}\Sigma_{23} \neq \Sigma_{11}^{-1}\Sigma_{13}$, we also have $E(X_1|X_2)|X_3] \neq E[E(X_2|X_1)|X_3]$.

(c) Assume that X_1, X_2, X_3 are i.i.d. and that $E[|X_1|] < \infty$.

$$E[X_1|X_1 + X_2 + X_3] = \frac{X_1 + X_2 + X_3}{3}$$

Since X_1, X_2, X_3 are i.i.d., $E[X_1|X_1+X_2+X_3] = E[X_2|X_1+X_2+X_3] = E[X_3|X_1+X_2+X_3]$. Thus

$$\begin{aligned} E[X_1|X_1 + X_2 + X_3] &= \frac{\sum_{i=1}^{3} E[X_i|X_1 + X_2 + X_3]}{3} \\ &= \frac{E[X_1 + X_2 + X_3|X_1 + X_2 + X_3]}{3} \\ &= \frac{X_1 + X_2 + X_3}{3}, \end{aligned}$$

where the last inequality follows since $X_1 + X_2 + X_3$ is measurable with respect to the σ-field generated by itself.