

BIOS 760: Midterm I 2010

1. Let $X = (X_1, X_2)$ be a bivariate random variable with density function

$$f(x_1, x_2) = 2, \quad 0 < x_1 < x_2 < 1.$$

- (a) (3 points) Write the cumulative distribution function of X .
- (b) (2 points) Write the marginal density of X_1 and X_2 .
- (c) (3 points) Are X_1 and X_2 independent? Explain.
- (d) (3 bonus points) Find $f(x_1|x_2)$ and compute the conditional expectation $E(X_1|X_2)$.

2. Let $X = (X_1, X_2)$ be as in Question 1.

Define the function $u(x_1, x_2) = \left(\frac{x_1}{x_2}, x_2\right)$ and denote $Y = (Y_1, Y_2) = u(X_1, X_2)$.

- (a) (3 points) Find the Jacobian of the function u .
- (b) (3 points) Find the density of Y .
- (c) (3 points) Are Y_1 and Y_2 independent? Explain.

3. Let X_1, X_2, X_3 be random variables on some probability space.

- (a) (4 points) Prove that

$$E[X_1 E(X_2|X_3)] = E[X_2 E(X_1|X_3)]$$

- (b) (4 points) Give an example where

$$E[E(X_1|X_2)|X_3] \neq E[E(X_2|X_1)|X_3]$$

Hint: Consider a multivariate normal vector (X_1, X_2, X_3) .

- (c) (3 bonus points) Assume that X_1, X_2, X_3 are i.i.d. and that $E[|X_1|] < \infty$. Show that

$$E[X_1|X_1 + X_2 + X_3] = \frac{X_1 + X_2 + X_3}{3}$$

Hint: Show first that $E[X_i|X_1 + X_2 + X_3]$, $i = 1, 2, 3$ are equal.