BIOS760: 2009 FALL SEMESTER MIDTERM EXAM II

Assume that $(X_i, Y_i), i = 1, ..., n$, are i.i.d observations satisfying

$$E[|X_i|^6] < \infty, \quad Y_i = \beta X_i + \epsilon_i,$$

where $\epsilon_1, ..., \epsilon_n$ are i.i.d with mean zeros and finite fourth moments and they are independent of $X_1, ..., X_n$. Let $\hat{\beta}$ be the least square estimator of β, i.e.,

$$\hat{\beta} = \frac{\sum_{i=1}^n X_i Y_i}{\sum_{i=1}^n X_i^2}.$$

1. (5 points) Show that $\hat{\beta} \to_{a.s.} \beta$.
2. (5 points) Derive the asymptotic distribution of $\hat{\beta}$ after a proper normalization.
3. (5 points) Let
$$T_n = \frac{1}{n}\sum_{i=1}^n (Y_i - \hat{\beta} X_i)^2 - \frac{1}{n}\sum_{i=1}^n (Y_i - \beta X_i)^2.$$
Find the asymptotic distribution of T_n after a proper normalization.
4. Now assume that β is known to be non-negative a priori. We define a new estimator $\tilde{\beta} = \max(\hat{\beta}, 0)$.
   (a) (5 points) Show that $\tilde{\beta} \to_{a.s.} \beta$.
   (b) (5 points) If $\beta > 0$, derive the asymptotic distribution of $\tilde{\beta}$ after a proper normalization.
   (c) (5 points) If $\beta = 0$, what is the asymptotic distribution of $\tilde{\beta}$ after a proper normalization?
5. (**This is a take-home question due 11 A. M. November 11th**). We want to obtain the following conditional central limit theorem: condition on $X_1, X_2, ...$ (i.e., $X_1, X_2, ..$ are held as fixed), $\sqrt{n}(\hat{\beta} - \beta)$ converges to the same limiting distribution of (2). To prove it, please verify the following steps.
   (a) (5 points) Let $F(\cdot)$ be the distribution function of $|X_1|$. Show
$$1 - F(x) \leq \frac{E[X_1^6]}{x^6}$$
   using the Markov's inequality.
   (b) (5 points) For any constant $\delta > 0$, write down the probability $P(\max_{i=1}^n |X_i| \geq \delta\sqrt{n})$ in terms of F and using the previous result to show
$$\sum_{n=1}^{\infty} P(\max_{i=1}^n |X_i| \geq \delta\sqrt{n}) < \infty.$$
   (c) (2 points) Cite the name of one lemma/theorem to conclude $\max_{i=1}^n |X_i|/\sqrt{n} \to 0$ almost surely.
   (d) (8 points) Use the previous results to verify that the Lindeberg-Feller condition holds, then conclude the conditional central limit theorem for $\sqrt{n}(\hat{\beta} - \beta)$.