BIOS760: 2009 FALL SEMESTER MIDTERM EXAM I

1. Suppose that Y is a one-dimensional random variable and X is a k-dimensional random vector. The joint distribution of Y and X is a multivariate normal distribution with mean zeros and covariance

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \quad \Sigma > 0.$$

   (a) (5 points) Find a $k \times 1$ constant vector γ such that $Var(Y - \gamma^T X)$ is the smallest. Denote such a vector as γ_m. Calculate $\sigma_m^2 = Var(Y - \gamma_m^T X)$.

   (b) (5 points) Derive the distribution of

$$\frac{(Y - \gamma_m^T X)^2 / \sigma_m^2}{X^T \Sigma_{22}^{-1} X}.$$

2. Let $(\Omega, \mathcal{A}, P)$ be a probability measure space. Suppose that Y is a random variable defined on this space with $Y(\Omega) = R$; that is, Y maps Ω to the whole real line. Define a class

$$\mathcal{C} = \left\{ Y^{-1}(B) : B \in \mathcal{B} \text{ where } \mathcal{B} \text{ is the Borel } \sigma\text{-field in } R \right\}.$$

   (a) (5 points) Show that $\mathcal{C}$ is a sub-σ-field of $\mathcal{A}$.

   (b) (5 points) For any set $Y^{-1}(B)$ in $\mathcal{C}$, we define a set function

$$\mu(Y^{-1}(B)) = \lambda(B),$$

   where λ is the Lebesgue measure in R. Show that μ is a measure in $(\Omega, \mathcal{C})$.

   (c) (5 points) From (a) and (b), both P and μ are measures in $(\Omega, \mathcal{C})$. Suppose that Y is a continuous random variable with density function $f(y)$. Show that P is dominated by μ.

   (d) (Bonus: 5 points) For (c), find the Randon-Nikodym derivative of $dP(\omega)/d\mu$.