Solution to BIOS760 Midterm 2008

1. (a) $\sum_{y=0}^{\infty} e^{-ay} = \sum_{y=0}^{\infty}(e^{-a})^y = (1-e^{-a})^{-1}$. Thus $c_1 = 1-e^{-a}$. Since $\int_0^1 e^{-au}du = a^{-1}(1-e^{-a})$, $c_2 = a(1-e^{-a})^{-1}$.

   i. Note that

$$\begin{aligned} P(X \le x) &= P(Y > \lfloor x \rfloor, X \le x) + P(Y = \lfloor x \rfloor, X \le x) + P(Y < \lfloor x \rfloor, X \le x) \\ &= b_1 + b_2 + b_3. \end{aligned}$$

   Since $P(Y > \lfloor x \rfloor) = P(Y \ge \lfloor x \rfloor + 1)$ and $P(U = 0) = 0$, we have $b_1 = 0$. Since $P(Y < \lfloor x \rfloor, X \le x) = P(Y \le \lfloor x \rfloor - 1, U \le 1)$ and $P(U \le 1) = 1$, $b_3 = P(Y < \lfloor x \rfloor)$. Finally, since $b_2 = P(Y = \lfloor x \rfloor, X \le x) = P(Y = \lfloor x \rfloor, Y + U \le x) = P(Y = \lfloor x \rfloor, U \le x - \lfloor x \rfloor)$, the desired conclusion follows.

   ii. By independence of U and Y, $P(Y = \lfloor x \rfloor, U \le x - \lfloor x \rfloor) = I\{x \ge 0\} c_1 e^{-a\lfloor x \rfloor} c_2 \times \int_0^{x-\lfloor x \rfloor} e^{-au}du = I\{x \ge 0\} c_1 c_2 e^{-a\lfloor x \rfloor} a^{-1}(1 - e^{-ax + a\lfloor x \rfloor}) = I\{x \ge 0\}\left(e^{-a\lfloor x \rfloor} - e^{-ax}\right)$. Since also

$$\begin{aligned} P(Y < \lfloor x \rfloor) &= c_1 I\{x \ge 1\} \sum_{y=0}^{\lfloor x \rfloor - 1} e^{-ay} \\ &= c_1 I\{x \ge 0\} \frac{1 - e^{-\lfloor x \rfloor}}{1 - e^{-a}} \\ &= I\{x \ge 0\}\left(1 - e^{-a\lfloor x \rfloor}\right), \end{aligned}$$

   we obtain that $P(X \le x) = I\{x \ge 0\}(1 - e^{-ax})$.

   iii. This is the exponential distribution.

2. (a) The field $\mathcal{C}$ consists of all possible sets obtained from finite set operations on A_1, A_2 and A_3. Since these sets are disjoint, we obtain that

$$\mathcal{C} = \{\emptyset, A_1, A_2, A_3, A_1 \cup A_2, A_1 \cup A_3, A_2 \cup A_3, R\}.$$

   Since a finite field is a σ-field because it is closed under countable set operations, $\mathcal{C}$ is also a σ-field.

   (b) Since Y is a simple function of sets in $\mathcal{C}$, it is measurable with respect to $\mathcal{C}$.

   (c) The conditional expectation $E[X|\mathcal{C}]$ is defined as the unique quantity that is (i) measurable with respect to $\mathcal{C}$ and (ii) satisfies $\int_G E[X|\mathcal{C}]dP = \int_G XdP$ for all $G \in \mathcal{C}$.

That Y satisfies (i) follows from part (b) above. For (ii), if we can find μ_j such that $\int_{A_j} \mu_j I_{A_j} dP = \int_{A_j} X dP$, we are done. But this just requires

$$\mu_j = \frac{\int_{A_j} X dP}{\int_{A_j} dP},$$

which is well defined for the normal distribution.

(d) Since X is standard normal, we have for any $0 \le a \le b \le \infty$,

$$\int_a^b x dP = \frac{1}{\sqrt{2\pi}} \int_a^b x e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{a^2/2}^{b^2/2} e^{-u} du = \frac{e^{-a^2/2} - e^{-b^2/2}}{\sqrt{2\pi}}.$$

Thus, letting Φ be the cumulative distribution function for a standard normal, we have

$$\begin{aligned} \mu_1 &= \frac{-1}{\sqrt{2\pi}\Phi(0)} = \frac{-2}{\sqrt{2\pi}}, \\ \mu_2 &= \frac{1 - e^{-1/2}}{\sqrt{2\pi}(\Phi(1) - \Phi(0))}, \\ \mu_3 &= \frac{e^{-1/2}}{\sqrt{2\pi}(1 - \Phi(1))}. \end{aligned}$$

3. (a) By Hölder's inequality, we have for any $p > 1$ and $a = p/(p-1)$ that

$$\int |fgh| d\mu \le \left(\int |f|^p d\mu \right)^{1/p} \left(\int |gh|^a d\mu \right)^{1/a}.$$

Also, for $s = q(p-1)/p$ and $v = r(p-1)/p$, we have by reapplication of Hölder's inequality that

$$\int |gh|^a d\mu \le \left(\int |g|^{as} d\mu \right)^{1/s} \left(\int |h|^{av} d\mu \right)^{1/v},$$

since

$$\frac{1}{s} + \frac{1}{v} = \frac{p}{q(p-1)} + \frac{p}{r(p-1)} = \frac{p}{p-1}\left(\frac{1}{q} + \frac{1}{r} \right) = 1.$$

The desired result now follows since $as = q$ and $av = r$.

(b) Using the above with $p = 2$ and $q = r = 4$, we obtain that

$$E|X_n Y_n Z_n| \le \left(E[X_n^2] \right)^{1/2} \left(E[Y_n^4] \right)^{1/4} \left(E[Z_n^4] \right)^{1/4}.$$

The desired result now follows.