BIOS 760 Midterm 2008

1. Let U and Y be independent, real random variables and let $0 < a < \infty$ be a constant. Assume Y has a density with respect to counting measure on the set of non-negative integers with probability mass function $p_Y(y) = c_1 e^{-ay}$ and that U has Lebesgue density $f_U(u) = I\{0 \le u \le 1\} c_2 e^{-au}$, where $0 < c_1, c_2 < \infty$ are constants. Please do the following:

   (a) (3 points) Show that $c_1 = 1 - e^{-a}$ and $c_2 = a(1 - e^{-a})^{-1}$.

   (b) Derive the cumulative distribution function F_X of $X = Y + U$ using the following steps:

      i. (2 points) Show that $P(Y + U \le x) = P(Y = \lfloor x \rfloor, U \le x - \lfloor x \rfloor)$ $+ P(Y < \lfloor x \rfloor)$, where $\lfloor x \rfloor$ is the greatest integer $\le x$.

      ii. (4 points) Derive $F_X(x) = P(Y + U \le x)$.

      iii. (1 point) What is the name of this distribution?

2. Let X be a real random variable on the probability space $(R, \mathcal{B}, P)$, where R is the set of real numbers, $\mathcal{B}$ is the Borel σ-field on R, and P is the standard normal probability measure (in other words, X has a standard normal distribution). Define the sets $A_1 = (-\infty, 0]$, $A_2 = (0, 1]$ and $A_3 = (1, \infty)$. Do the following:

   (a) (3 points) Construct the field $\mathcal{C}$ generated by A_1, A_2 and A_3 viewed as subsets of R. Argue briefly why $\mathcal{C}$ is also a σ-field.

   (b) (2 points) Show that the random quantity $Y = \sum_{j=1}^{3} \mu_j I_{A_j}(X)$ is measurable with respect to $\mathcal{C}$ for any choice of real constants μ_1, μ_2 and μ_3.

   (c) (2 points) Show that the conditional expectation $E[X|\mathcal{C}]$ has the form Y.

   (d) (3 points) Derive the values of μ_1, μ_2 and μ_3 that make Y equal to $E[X|\mathcal{C}]$.

3. Let X_n, Y_n and Z_n be sequences of real random variables (not necessarily independent) which satisfy $\lim_{n\to\infty} E[X_n^2] = 0$, $\limsup_{n\to\infty} E[Y_n^4] < \infty$ and $\limsup_{n\to\infty} E[Z_n^4] < \infty$. Do the following:

   (a) (5 points) Show that for any measurable functions f, g, h and measure μ,

   $$\int |fgh| d\mu \le \left(\int |f|^p d\mu\right)^{1/p} \left(\int |g|^q\right)^{1/q} \left(\int |h|^r\right)^{1/r},$$

   for any $p, q, r > 1$ with $1/p + 1/q + 1/r = 1$.

   (b) (5 points) Show that $\lim_{n\to\infty} E|X_n Y_n Z_n| = 0$.