Solution to BIOS760 Midterm 2007

1. (a) Note

$$Cov(Z_1 - \rho Z_2, Z_2) = 0.$$

Thus, $Z_1 - \rho Z_2$ is independent of Z_2. Moreover, since $Z_2 \sim N(0, \sigma^2)$ and

$$Z_1 - \rho Z_2 \sim N(0, (1-\rho^2)\sigma^2).$$

Hence,

$$\frac{Z_2}{\sqrt{(Z_1 - \rho Z_2)^2}} \sim \frac{1}{\sqrt{1-\rho^2}} t(1).$$

(b) The density is

$$\exp\left\{ -\frac{Z_1^2}{2(1-\rho^2)\sigma^2} + \frac{Z_1 Z_2 \rho}{(1-\rho^2)\sigma^2} - \frac{Z_2^2}{2(1-\rho^2)\sigma^2} - \log(\sqrt{1-\rho^2}\sigma^2) \right\} /2\pi.$$

It is a 2-parameter exponential family. Let

$$\eta_1 = -\frac{1}{2(1-\rho^2)\sigma^2}, \quad \eta_2 = \frac{\rho}{(1-\rho^2)\sigma^2}.$$

Then the density can be rewritten as

$$\exp\left\{ \eta_1 (Z_1^2 + Z_2^2) + \eta_2 Z_1 Z_2 + \frac{1}{2}\log(4\eta_1^2 - \eta_2^2) \right\} /2pi.$$

(c) Treating η_2 as the only parameter, we obtain the moment generating function for $Z_1 Z_2$ is

$$\exp\left\{ -\frac{1}{2}\log(4\eta_1^2 - (\eta_2 + t)^2) + \frac{1}{2}\log(4\eta_1^2 - \eta_2^2) \right\} = \sqrt{\frac{4\eta_1^2 - \eta_2^2}{4\eta_1^2 - (\eta_2 + t)^2}}.$$

Its second derivative at $t = 0$ is

$$(1 + 2\rho^2)\sigma^4.$$

Thus, $E[Z_1^2 Z_2^2]$ is $(1 + 2\rho^2)\sigma^4$.

2. (a) Note $\lambda_F(\{0\}) = 0$ but $\lambda_G(\{0\}) = 1$ and $\lambda_F((0,1]) = 1 - e^{-1}$ but $\lambda_G((0,1]) = 0$.

(b) Clearly $\mu(\emptyset) = 0$ and $\mu(R) = 1$. Since λ_F and λ_G both satisfy countable additivity, so is μ.

(c) For any $x_0 > 0$,

$$\{x : f(x) \leq x_0\} = \{x : x \leq 2\log x_0\} \in \mathcal{B};$$

for any $x_0 \leq 0$, $\{x : f(x) \leq x_0\} = \emptyset \in \mathcal{B}$. Thus, $f(x)$ is a random variable.

$$E[f] = \frac{1}{2}\int f(x)d\lambda_F(x) + \frac{1}{2}\int f(x)d\lambda_G(x)$$

$$= \frac{1}{2}\int e^{x/2}e^{-x}I(x \geq 0)dx + \frac{1}{2}f(0) = \frac{3}{2}.$$

3. (a) For any $\epsilon > 0$,

$$P(|X_n| > \epsilon) \leq P(|X| > n) \leq \frac{E[|X|]}{n} \to 0.$$

(b) Since $|X_n| \leq |X|$ and $X_n \to_p 0$, by DCT, $E[X_n] \to 0$.

(c) Since $n|X_n| = n|X|I(|X| > n) \leq |X|^2 I(|X| > n) \leq |X|^2$ and the same argument as in (a) gives $nX_n \to_p 0$, by DCT, $nE[X_n] \to 0$.