BIOS760 Midterm 2007

1. Suppose $(Z_1, Z_2)'$ follows a bivariate normal distribution with mean zeros and covariance

$$\Sigma = \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix},$$

where $\rho \in (-1, 1)$ and $\sigma > 0$.

   (a) (6 points) Derive the distribution of

$$\frac{Z_2}{\sqrt{(Z_1 - \rho Z_2)^2}}.$$

   (b) (6 points) Consider ρ and σ^2 are parameters. Show that the distribution of $(Z_1, Z_2)'$ forms an exponential family. Find a canonical expression via appropriate reparameterization.

   (c) (6 points) Calculate $E[Z_1^2 Z_2^2]$.

2. Suppose $\mathcal{B}$ is the Borel σ-field in R. Define

$$F(x) = (1 - e^{-x})I(x \geq 0), \quad G(x) = I(x \geq 0).$$

Let λ_F and λ_G be the Lebesgue-Stieljes measures generated by F and G respectively.

   (a) (6 points) Show that λ_F is not dominated by λ_G and neither is λ_G dominated by λ_F.

   (b) (6 points) For any $B \in \mathcal{B}$, we define

$$\mu(B) = \frac{1}{2}\lambda_F(B) + \frac{1}{2}\lambda_G(B).$$

   Show that μ is a probability measure in R.

   (c) (10 points) For any $x \in R$, we define

$$f(x) = e^{x/2}.$$

   Show f is a random variable in the measure space $(R, \mathcal{B}, \mu)$ and calculate $E[f] \equiv \int f d\mu$.

3. Let X be a random variable defined on a probability measure space $(\Omega, \mathcal{A}, P)$ satisfying $E[|X|] < \infty$. Define $X_n(\omega) = X(\omega)I(|X(\omega)| > n)$ for any $\omega \in \Omega$; i.e., $X_n(\omega) = X(\omega)$ if $|X(\omega)| > n$ and is zero otherwise.

   (a) (6 points) Show $X_n \to_p 0$.

   (b) (6 points) Show $E[X_n] \to 0$.

   (c) (8 points) If we further assume $E[X^2] < \infty$, show $nE[X_n] \to 0$.