Solution to BIOS760 Midterm Exam, Fall 2006

1. (a) One simple construction is to consider $(R^2, \mathcal{B}\times\mathcal{B}, \lambda_F)$, where λ_F is the Lebesgue-Stieljes measure generated by F in R^2 and F is the joint distribution of X and Y. Then define $(X(\omega_1,\omega_2), Y(\omega_1,\omega_2)) = (\omega_1,\omega_2)$. The other construction is to define a measure space $(R, \mathcal{B}, \lambda_{F_X})$, where λ_{F_X} is the Lebesgue-Stieljes measure generated by the distribution of X; similarly, define another measure space $(R, \mathcal{B}, \lambda_{F_Y})$, where λ_{F_Y} is the Lebesgue-Stieljes measure generated by the distribution of Y. Then consider the product of these two measure spaces $(R\times R, \mathcal{B}\times\mathcal{B}, \lambda_{F_X}\times\lambda_{F_Y})$ and on this space, define $(X(\omega_1,\omega_2), Y(\omega_1,\omega_2)) = (\omega_1,\omega_2)$.

   (b) $F_Z(z) = 0.5zI(0 \le z < 1) + 0.5$. The Z-induced measure is the Lebesgue-Stieljes measure generated by F_Z.

   (c) F_Z is continuous except a positive jump at 1. Thus, a dominating measure is $\lambda + \mu^{\#}$ where λ is the Lebesgue measure and $\mu^{\#}$ is the counting measure on $\{1\}$. The density function is

$$f_Z(z) = 0.5I(z \in [0,1)) + 0.5I(z = 1).$$

2. (a) Consider the characteristic function of X_n:

$$\phi_n(t) = E[e^{it(1-U/n^\alpha)^n}] = \{e^{it} + e^{it(1-1/n^\alpha)^n}\}/2.$$

   When $\alpha = 0$, $X_n = (1-U) \to_d U$. When $0 < \alpha < 1$, $\phi_n(t) \to_p (1+e^{it})/2$. Thus, $X_n \to_d U$. When $\alpha = 1$, $X_n \to_{a.s.} \exp\{-U\}$. When $\alpha > 1$, $\phi_n(t) \to e^{it}$ so $X_n \to_d 1$.

   (b) Since U is between 0 and 1 almost surely, $(1-U/n^\alpha)^n$ converges to 0 if $\alpha < 1$, e^{-U} if $\alpha = 1$, and 1 if $\alpha > 1$. Then we conclude that the limit distribution of X_n is 0 if $\alpha < 1$, e^{-U} if $\alpha = 1$, and 1 if $\alpha > 1$.

3. (a) Consider $X_n = X$ and $Y_n = -X$ where X has $N(0,1)$ distribution. Then $X_n \to_d \tilde{X}$ and $Y_n \to_d X$ where $\tilde{X}$ is an i.i.d copy of X. Clearly, $X_n + Y_n = 0$ does not converge in distribution to $\tilde{X} + X \sim N(0,2)$.

   (b) Use the characteristic function. In fact, we can even show $(X_n, Y_n) \to_d (X, Y)$.

   (c) First show $(A_n, B_n) \to_p (a, b)$. Apply the Slutsky's theorem and we obtain

$$A_nX_n + B_nY_n = (A_n, B_n)\begin{pmatrix} X_n \\ Y_n \end{pmatrix} \to_d (a,b)\begin{pmatrix} X \\ Y \end{pmatrix} = aX + bY.$$