1. (a) (15 points) Construct a probability measure space \((\Omega, \mathcal{A}, P)\) and define a bivariate random vector \((X, Y)\) on this space satisfying

\[
X \sim \text{Uniform}(0, 1), \quad Y \sim \text{Bernoulli}(0, 1, 0.5), \quad \text{and } X \text{ and } Y \text{ are independent.}
\]

Your answer should include the specific details on the measure space and how \(X\) and \(Y\) are defined. Note that there are many ways for this construction but you only need to show one.

(b) (10 points) Suppose \(X\) and \(Y\) are already defined in (a). We define another random variable \(Z\):

\[
Z = \begin{cases} 
1, & Y = 0, \\
X, & Y = 1
\end{cases}
\]

What is the cumulative distribution function of \(Z\)? What is the induced measure by \(Z\)?

(c) (15 points) Provide a dominating measure \(\mu\) depending on the Lebesgue measure and the counting measure, so that \(Z\) has a density function with respect to \(\mu\). Give the expression of \(Z\)'s density function.

2. (a) (10 points) Let \(U\) be a Bernoulli variable with \(P(U = 0) = P(U = 1) = 1/2\). Define

\[
X_n = \left(1 - \frac{U}{n^\alpha}\right)^n
\]

where \(\alpha \geq 0\). Can you find the limiting distribution of \(X_n\)? If so, specify the limiting distribution and justify it. (Hint: consider \(\alpha = 0, 0 < \alpha < 1, \alpha = 1\) and \(\alpha > 1\).)

(b) (10 points) Now assume \(U\) from Uniform(0, 1). What is the answer to (a)?

3. Suppose that \(X_n\) and \(Y_n\) are two sequences of random variables defined on a probability measure space. Assume \(X_n \to_d X\) and \(Y_n \to_d Y\).

(a) (10 points) Give an example of \(X_n\) and \(Y_n\) satisfying the assumption but \((X_n + Y_n)\) does not converge in distribution to \((X + Y)\).

(b) (15 points) Show that if \(X_n\) and \(Y_n\) are independent, then \((X_n + Y_n) \to_d (X + Y)\).

(c) (15 points) Let \(A_n\) and \(B_n\) be another two sequences of random variables defined on the same probability measure space and \(A_n \to_p a\) and \(B_n \to_p b\) with \(a\) and \(b\) being constants.

Show that if \(X_n\) and \(Y_n\) are independent, then \((A_nX_n + B_nY_n) \to_d (aX + bY)\).