BIOS760 Midterm Exam, Fall 2006

1. (a) (15 points) Construct a probability measure space $(\Omega, \mathcal{A}, P)$ and define a bivariate random vector (X, Y) on this space satisfying

$$X \sim \text{Uniform}(0,1), \quad Y \sim \text{Bernoulli}(0,1,0.5), \quad \text{and } X \text{ and } Y \text{ are independent.}$$

Your answer should include the specific details on the measure space and how X and Y are defined. Note that there are many ways for this construction but you only need to show one.

(b) (10 points) Suppose X and Y are already defined in (a). We define another random variable Z:

$$Z = \begin{cases} 1, & Y = 0, \\ X, & Y = 1 \end{cases}$$

What is the cumulative distribution function of Z? What is the induced measure by Z?

(c) (15 points) Provide a dominating measure μ depending on the Lebesgue measure and the counting measure, so that Z has a density function with respect to μ. Give the expression of Z's density function.

2. (a) (10 points) Let U be a Bernoulli variable with $P(U=0) = P(U=1) = 1/2$. Define

$$X_n = \left(1 - \frac{U}{n^\alpha}\right)^n$$

where $\alpha \geq 0$. Can you find the limiting distribution of X_n? If so, specify the limiting distribution and justify it. (Hint: consider $\alpha = 0$, $0 < \alpha < 1$, $\alpha = 1$ and $\alpha > 1$.)

(b) (10 points) Now assume U from Uniform$(0,1)$. What is the answer to (a)?

3. Suppose that X_n and Y_n are two sequences of random variables defined on a probability measure space. Assume $X_n \to_d X$ and $Y_n \to_d Y$.

(a) (10 points) Give an example of X_n and Y_n satisfying the assumption but $(X_n + Y_n)$ does not converge in distribution to $(X + Y)$.

(b) (15 points) Show that if X_n and Y_n are independent, then $(X_n + Y_n) \to_d (X + Y)$.

(c) (15 points) Let A_n and B_n be another two sequences of random variables defined on the same probability measure space and $A_n \to_p a$ and $B_n \to_p b$ with a and b being constants. Show that if X_n and Y_n are independent, then $(A_n X_n + B_n Y_n) \to_d (aX + bY)$.