Solution to Midterm Exam

1. (a) Clearly, $\nu(A) \geq 0$ and $\nu(\emptyset) = 0$. For any disjoint and countably many sets $B_1, B_2, ...$ in $\mathcal{A}$,

$$\nu(\cup_{k=1}^{\infty} B_k) = \sum_{i=1}^{n} \omega_i \mu(\cup_{k=1}^{\infty}(B_k \cap A_i)) = \sum_{i=1}^{n} \omega_i \left\{ \sum_{k=1}^{\infty} \mu(B_k \cap A_i) \right\}$$
$$= \sum_{k=1}^{\infty} \left\{ \sum_{i=1}^{n} \omega_i \mu(B_k \cap A_i) \right\} = \sum_{k=1}^{\infty} \nu(B_k).$$

Thus, ν is a measure.

(b) Clearly, if $\mu(A) = 0$ then $\nu(A) = 0$. Thus, $\nu \prec\prec \mu$.

(c) Note $\nu(A) = \int_A \left\{ \sum_{i=1}^{n} \omega_i I_{A_i}(x) \right\} d\mu(x)$. Then

$$\frac{d\nu}{d\mu}(x) = \sum_{i=1}^{n} \omega I_{A_i}(x).$$

2. Since $|(1 + t/n)^{-n}| \leq (1 + t/2)^{-2}$, by the dominated convergence theorem,

$$\lim_{n\to\infty} \int_{t\in[0,\infty)} (1 + t/n)^{-n} d\lambda(t) = \int_{t\in[0,\infty)} \lim_{n\to\infty} (1 + t/n)^{-n} d\lambda(t) = \int_{t\in[0,\infty)} e^{-t} dt = 1.$$

From the Fubini-Tonelli theorem,

$$\int_{(t,x)\in[0,\infty)\times(1,2)} \frac{\exp\{-xt\}}{1 + x^2} d(\lambda \times \lambda)(t, x) = \int_{x\in(1,2)} \int_{t\in[0,\infty)} \frac{\exp\{-xt\}}{1 + x^2} d\lambda(t) d\lambda(x)$$
$$= \int_1^2 \frac{1}{x(1 + x^2)} dx = \int_1^4 \frac{1}{2y(1 + y)} dy = \frac{1}{2} \log(8/5).$$

3. (a) Since $Cov(X + Y, X - Y) = Cov(X, X) - Cov(Y, Y) = 0$, $X + Y$ and $X - Y$ are independent. Clearly, $R^2 = X^2 + Y^2 = [(X + Y)^2 + (X - Y)^2]/2$.

(b) Given $X - Y = 0$, $R^2 = (X + Y)^2/2$. Moreover, from (a), the conditional distribution of R^2 given $X - Y = 0$ is the same as the unconditional distribution of R^2. Since $(X + Y)/\sqrt{2}$ has a standard normal distribution, R^2 follows a chi-squared distribution with one degree of freedom.

(c) Using the transformation $X = R\cos\Theta, Y = R\sin\Theta$, we obtain the joint density of (R, Θ) as

$$r \exp\{-r^2/2\} I(r \geq 0) \times \frac{1}{2\pi} I_{[0,2\pi)}(\theta).$$

Thus, R and Θ are independent. Moreover, R has the density function $r\exp\{-r^2/2\}$ $I(r \geq 0)$ then

$$P(R^2 \leq r) = \exp\{-r/2\}.$$

That is, R^2 has a distribution $\Gamma(1,2)$; i.e., R^2 follows a chi-squared distribution with two degrees of freedom.

(d) This is because the two sub-σ-fields are different: one is the σ-field generated by $\{X - Y = 0\}$; the other one is the σ-field generated by $\{\Theta = \pi/4\}$ or the σ-field generated by $\{\Theta = 5\pi/4\}$.

4. " $\Rightarrow$": If $X_n \to_p X$, then for any small $\epsilon > 0$,

$$P(\frac{|X_n - X|}{1 + |X_n - X|} > \epsilon) = P(|X_n - X| > \epsilon/(1-\epsilon)) \to 0.$$

Thus,

$$\frac{|X_n - X|}{1 + |X_n - X|} \to_p 0.$$

From the dominated convergence theorem, since $|X_n - X|/(1 + |X_n - X|) \leq 1$, we obtain

$$E\left[\frac{|X_n - X|}{1 + |X_n - X|}\right] \to 0.$$

" $\Leftarrow$": Since $g(x) = x/(1+x)$ is increasing, by the Markov's inequality,

$$P(|X_n - X| > \epsilon) \leq \frac{E[g(|X_n - X|)]}{g(\epsilon)} \to 0.$$