BIOS260 Midterm Exam

Oct. 07, 2004

1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Suppose $A_1, ..., A_n \in \mathcal{A}$ and they are a partition of Ω; i.e., $A_1, ..., A_n$ are disjoint and $\cup_{i=1}^n A_i = \Omega$. Given any non-negative constants $\omega_1, ..., \omega_n$, we define

$$\nu(A) = \sum_{i=1}^n \omega_i \mu(A \cap A_i), \quad A \in \mathcal{A}.$$

   (a) (4 *points*) Show that ν is a measure.

   (b) (2 *points*) Show $\nu \prec\prec \mu$.

   (c) (4 *points*) Calculate the Randon-Nikodym derivative of ν with respect to μ.

2. (10 *points*) Calculate the following expressions

$$\lim_{n\to\infty} \int_{t\in[0,\infty)} (1+t/n)^{-n} d\lambda(t), \quad \int_{(t,x)\in[0,\infty)\times(1,2)} \frac{\exp\{-xt\}}{1+x^2} d(\lambda \times \lambda)(t,x),$$

where λ is the Lebesgue measure. Justify your answers.

3. Let X and Y be independent, each having the standard normal distribution, and let (R, Θ) be the polar coordinates of (X, Y). That is, $X = R\cos\Theta, Y = R\sin\Theta$ for $R \in [0,\infty), \Theta \in [0, 2\pi)$.

   (a) (5 *points*) Show that $X+Y$ and $X-Y$ are independent and that $R^2 = [(X+Y)^2 + (X-Y)^2]/2$.

   (b) (5 *points*) Conclude that the conditional distribution of R^2 given $X - Y = 0$ is the chi-squared distribution with one degree of freedom.

   (c) (5 *points*) Obtain the joint distribution of (R, Θ) and show that the conditional distribution of R^2 given Θ is chi-squared with two degrees of freedom.

   (d) (5 *points*) From (b), when $X - Y = 0$, the conditional distribution of R^2 is chi-squared with one degree of freedom. From (c), if $\Theta = \pi/4$ or $\Theta = 5\pi/4$, the conditional distribution of R^2 is chi-squared with two degrees of freedom. But the events $\{X - Y = 0\}$ and $\{\Theta = \pi/4\} \cup \{\Theta = 5\pi/4\}$ are the same. Explain the apparent contradiction.

4. (10 *points*) Suppose that X_n and X are random variables. Show that $X_n \to_p X$ if and only if

$$E\left[\frac{|X_n - X|}{1 + |X_n - X|}\right] \to 0.$$