BIOS760 Homework #2

Question 1: Lecture note, page 39, Problem 1.

Question 2: Lecture note, page 39, Problem 3.

Question 3: Lecture note, page 39, Problem 4.

Question 4: Let $(R, \mathcal{B}, \mu_F)$ be a measure space, where $\mathcal{B}$ is the Borel σ-filed and μ_F is the Lebesgue-Stieljes measure generated from

$$F(x) = \sum_{n=1}^{\infty} 2^{-n} I(x \geq n^{-1}) + (e^{-1} - e^{-x}) I(x \geq 1).$$

(a) For any interval $(a, b]$, calculate $\mu_F((a, b])$.

(b) Use the uniqueness of measure extension in the Carotheodory extension theorem to show

$$\mu_F(B) = \sum_{n=1}^{\infty} I(n^{-1} \in B) 2^{-n} + \int_B e^{-x} I(x \geq 1) d\lambda(x)$$

for any $B \in \mathcal{B}$, where λ is the Lebesgue measure.

(c) Show that for any measurable function X in $(R, \mathcal{B})$ with $\int |X| d\mu_F < \infty$,

$$\int X(x) d\mu_F(x) = \sum_{n=1}^{\infty} X(n^{-1}) 2^{-n} + \int X(x) e^{-x} I(x \geq 1) d\lambda(x).$$

Hint: use a sequence of simple functions to approximate X.

(d) Using the above result and the fact that for any Riemann integrable function, its Riemann integral is the same as its Lebesgue integral, calculate the integration $\int x^2 d\mu_F(x)$.

Question 5: Let $(R, \mathcal{B}, \lambda)$ be the Borel measure space with λ denoting the Lebesgue measure.

(a) Given any $B \in \mathcal{B}$ with $\lambda(B) < \infty$, show that for any $\epsilon > 0$, there exit a finite number of disjoint intervals, $(a_i, b_i], i = 1, ..., n$ such that

$$\lambda \left((B - \cup_{i=1}^{n} (a_i, b_i]) \cup (\cup_{i=1}^{n} (a_i, b_i] - B) \right) < \epsilon.$$

(Hint: use the construction in the Carotheodory extension theorem).

(b) For any integrable function f, show that for any $\epsilon > 0$, there exits a step function $g = \sum_{i=1}^{n} x_i I_{A_i}$, where $A_1, ..., A_n$ are disjoint and bounded intervals, such that

$$\int |f(x) - g(x)| d\lambda(x) < \epsilon.$$

(Hint: consider a simple function approximating f and use the first result).

(c) For any integrable function f, show that for any $\epsilon > 0$, there exits a continuous integrable function h with bounded support such that $\int |f(x) - h(x)| d\lambda(x) < \epsilon$. (Hint: construct a continuous function to approximate the simple function given in (b)).