Solution to 2009 FALL SEMESTER FINAl EXAM

Likelihood function

1(a) The likelihood function is

$$\prod_{i=1}^{m}(\lambda^{-1}e^{-X_i/\lambda}) \times (1-e^{-\tau/\lambda})^{n-m}.$$

Method of moments

2(a) Since $m = \sum_{i=1}^{n} I(X_i \geq \tau)$, m is a binomial distribution $Bin(n,p)$ where $p = e^{-\tau/\lambda}$.

2(b) $\hat{\lambda}_1 = -\tau/\log(m/n)$. Since $m/n \rightarrow_{a.s.} e^{-\tau/\lambda}$ by SLLN, $\hat{\lambda}_1 \rightarrow_{a.s.} \lambda$. Moreover,

$$\sqrt{n}(m/n - e^{-\tau/\lambda}) \rightarrow_d N\left(0, e^{-\tau/\lambda}(1-e^{-\tau/\lambda})\right).$$

From the delta method,

$$\sqrt{n}(\hat{\lambda}_1 - \lambda) \rightarrow_d N\left(0, \lambda^4(e^{\tau/\lambda}-1)/\tau^2\right).$$

Complete data analysis

3(a) We sample data from the population whose levels are detectable; thus, the density should be conditioned on $X \geq \tau$. Then the likelihood function is

$$\prod_{i=1}^{m}(\lambda^{-1}e^{-(X_i-\tau)/\lambda}).$$

3(b) The complete sufficient statistic is $\sum_{i=1}^{m}(X_i-\tau)$, whose conditional mean is $\sum_{i=1}^{m} E[X_i - \tau|X_i \geq \tau] = m\lambda$. Thus, the UMVUE is

$$\hat{\lambda}_2 = \sum_{i=1}^{m}(X_i-\tau)/m.$$

Its conditional variance is λ^2/m.

3(c) The information from these data is m/λ^2. So the UMVUE attains the Cramer-Rao bound.

3(d) Note

$$\hat{\lambda}_2 - \lambda = \frac{\sum_{i=1}^{n}(X_i-\tau-\lambda)I(X_i \geq \tau)}{\sum_{i=1}^{n} I(X_i \geq \tau)}.$$

By the CLT,

$$n^{-1/2}\sum_{i=1}^{n}(X_i-\tau-\lambda)I(X_i \geq \tau) \rightarrow_d N(0, \lambda^2 e^{-\tau/\lambda}).$$

Additionally,

$$n^{-1}\sum_{i=1}^{n} I(X_i \geq \tau) \to_{a.s.} e^{-\tau/\lambda}.$$

Thus,

$$\sqrt{n}(\hat{\lambda}_2 - \lambda) \to_d N(0, \lambda^2 e^{\tau/\lambda}).$$

Maximum likelihood estimation

4(a) The E-step calculates that for $i \geq m+1$,

$$E[X_i|X_i < \tau, \lambda^{(k)}] = \frac{\lambda^{(k)} - \lambda^{(k)} e^{-\tau/\lambda^{(k)}} - \tau e^{-\tau/\lambda^{(k)}}}{1 - e^{-\tau/\lambda^{(k)}}}.$$

The M-step maximizes

$$\sum_{i=1}^{m}\{-\log\lambda - X_i/\lambda\} + \sum_{i=m+1}^{n}\left\{-\log\lambda - E[X_i|X_i < \tau, \lambda^{(k)}]/\lambda\right\} = 0.$$

Thus,

$$\lambda^{(k+1)} = \frac{1}{n}\left\{\sum_{i=1}^{m} X_i + \sum_{i=m+1}^{n} E[X_i|X_i < \tau, \lambda^{(k)}]\right\}.$$

4(b) The information is

$$I(\lambda) = -E\left[\frac{\partial^2}{\partial\lambda^2}\left\{I(X < \tau)\log(1 - e^{-\tau/\lambda}) + I(X \geq \tau)\log(e^{-X/\lambda}/\lambda)\right\}\right]$$

$$= \frac{\tau^2 e^{-\tau/\lambda}/\lambda^4}{1 - e^{-\tau/\lambda}} + e^{-\tau/\lambda}/\lambda^2.$$

By the MLE theory,

$$\sqrt{n}(\hat{\lambda}_3 - \lambda) \to_d N(0, I(\lambda)^{-1}).$$

4(c) An asymptotic 95%-confidence interval for λ based on $\hat{\lambda}_3$ is given by

$$\left(\hat{\lambda}_2 - 1.96/\sqrt{I(\hat{\lambda}_2)n}, \hat{\lambda}_2 + 1.96/\sqrt{I(\hat{\lambda}_2)n}\right).$$

4(d) They are respectively

$$\frac{1}{1 + \lambda^2(1 - e^{-\tau/\lambda})/\tau^2}$$

and

$$\frac{1}{\frac{\tau^2/\lambda^2}{1 - e^{-\tau/\lambda}} + 1}.$$