Solution to BIOS760 Final Exam, Fall 2007

1. (a) The observed likelihood function is given by

$$\prod_{i=1}^{n}\left\{(2\pi)^{-1/2}e^{-X_i}\exp\{-e^{-2X_i}(Y_i-\beta)^2/2\}f(X_i)\right\}.$$

(b) By direct calculation, the CR lower bound is given by

$$\frac{1}{\sum_{i=1}^{n}E[e^{-2X_i}]}=\frac{2}{(1-e^{-2})\sum_{i=1}^{n}i^{-1}}.$$

(c) Using the weighted CLT and noting $\omega_{ni}=i^{-1/2}$ satisfy

$$\sum_{i=1}^{n}\omega_{ni}^2\to\infty,\quad \frac{\max_i|\omega_{ni}|}{\sqrt{\sum_{i=1}^{n}\omega_{ni}^2}}\to 0,$$

we obtain

$$\frac{\sum_{i=1}^{n}i^{-1/2}e^{-U_i}\epsilon_i}{\sqrt{\sum_{i=1}^{n}i^{-1}(1-e^{-2})/2}}\to_d N(0,1).$$

(d) Note

$$Var\left(\frac{\sum_{i=1}^{n}i^{-1}e^{-2U_i}}{(1-e^{-2})/2\sum_{i=1}^{n}i^{-1}}\right)=\frac{Var(e^{-2U_1})\sum_i 1/i^2}{(1-e^{-2})^2/4(\sum_{i=1}^{n}i^{-1})^2}\to 0$$

and its mean is 1. The result follows from the Chebyshev's inequality.

(e) Since

$$\hat\beta-\beta=\frac{\sum_{i=1}^{n}i^{-1/2}e^{-U_i}\epsilon_i}{\sum_{i=1}^{n}i^{-1}e^{-2U_i}},$$

from the previous results, we obtain

$$\left\{\sum_{i=1}^{n}i^{-1}(1-e^{-2})/2\right\}^{-1/2}(\hat\beta-\beta)\to_d N(0,1).$$

2. (a) $X_{(1)}$ is a complete and sufficient statistic for θ.

(b) Since $E[X_{(1)}]=\theta+1/n$, the UMVUE is $X_{(1)}-1/n$.

(c) The MLE is $X_{(1)}$.

(d) $Var(\hat\theta_2)=1/n^2$ and $E[\hat\theta_2]=\theta+1/n$. The consistency follows from the Chebyshev's inequality.

(e) Since $P(\hat\theta_2-\theta_0\le x)=1-e^{-nx}$ for $x>0$, we obtain

$$n(\hat\theta_2-\theta_0)\to_d Exp(1).$$

The MLE theory fails since the support depends on θ.

(f) $n(\hat\theta_1-\theta_0)\to_d Exp(1)-1$.

3. (a) The observed likelihood function is

$$\prod_{i=1}^{n}\left\{\Phi\left(\frac{25-\beta_0-\beta_1X_i}{\sigma}\right)\right\}^{1-Y_i}\left\{1-\Phi\left(\frac{25-\beta_0-\beta_1X_i}{\sigma}\right)\right\}^{Y_i}.$$

(b) The complete likelihood function is

$$\prod_{i=1}^{n}\left\{\frac{1}{\sigma}\phi(\frac{Z_i-\beta_0-\beta_1 X_i}{\sigma})\right\}.$$

(c) The M-step is

$$\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}=\left(\sum_{i=1}^{n}\begin{pmatrix}1\\ X_i\end{pmatrix}(1\quad X_i)\right)^{-1}\begin{pmatrix}\sum_i E[Z_i|Y_i,X_i]\\ \sum_i X_i E[Z_i|Y_i,X_i]\end{pmatrix},$$

$$\sigma^2=\frac{1}{n}\sum_{i=1}^{n}\left\{E[Z_i^2|Y_i,X_i]-(\beta_0+\beta_1X_i)E[Z_i|Y_i,X_i]+(\beta_0+\beta_1X_i)^2\right\}.$$

The E-step calculates

$$E[Z_i|Y_i,X_i]=\begin{cases}\frac{\beta_0+\beta_1X_i-\sigma\phi(\frac{25-\beta_0-\beta_1X_i}{\sigma})}{\Phi(\frac{25-\beta_0-\beta_1X_i}{\sigma})}, & Y_i=0,\\ \frac{\beta_0+\beta_1X_i+\sigma\phi(\frac{25-\beta_0-\beta_1X_i}{\sigma})}{1-\Phi(\frac{25-\beta_0-\beta_1X_i}{\sigma})}, & Y_i=1.\end{cases}$$

$$E[Z_i^2|Y_i,X_i]=(\beta_0+\beta_1X_i)^2+2(\beta_0+\beta_1X_i)E[Z_i|Y_i,X_i]$$
$$+\begin{cases}\sigma^2(-\frac{25-\beta_0-\beta_1X_i}{\sigma}\phi(\frac{25-\beta_0-\beta_1X_i}{\sigma})/\Phi(\frac{25-\beta_0-\beta_1X_i}{\sigma})+1)/, & Y_i=0,\\ \sigma^2(\frac{25-\beta_0-\beta_1X_i}{\sigma}\phi(\frac{25-\beta_0-\beta_1X_i}{\sigma})/(1-\Phi(\frac{25-\beta_0-\beta_1X_i}{\sigma}))+1), & Y_i=1.\end{cases}$$