Solution to BIOS760 Final Exam, Fall 2006

1. (a) The likelihood function for $Y_1, ..., Y_n$ is given as

$$(2\pi\sigma^2)^{-n/2}\prod_{i=1}^n i^{1/2}\exp\left\{-\frac{\sum_{i=1}^n iY_i^2}{2\sigma^2}+\frac{\sum_{i=1}^n iY_i}{\sigma^2}\mu-\frac{\sum_{i=1}^n i}{2\sigma^2}\mu^2\right\}.$$

This is a 2-parameter exponential family with complete sufficient statistics

$$\left(\sum_{i=1}^n iY_i, \sum_{i=1}^n iY_i^2\right).$$

(b) Since $E[\sum_{i=1}^n iY_i] = \mu\sum_{i=1}^n i$, the UMVUE for μ is

$$2\frac{\sum_{i=1}^n iY_i}{n(n+1)}.$$

(c) The variance of the UMVUE is

$$2\frac{\sigma^2}{n(n+1)}.$$

While, the Fisher information matrix for μ and σ^2 is calculated as

$$\begin{pmatrix} \sum_{i=1}^n i/\sigma^2 & 0 \\ 0 & n/(2\sigma^4) \end{pmatrix}.$$

The CR-lower bound for μ is $\sigma^2/\sum_{i=1}^n i$. Thus, the UMVUE attains this bound.

2. (a) We need find $a_1, ..., a_n$ minimizing

$$Var(\sum_{i=1}^n a_iY_i) = \sigma^2\sum_{i=1}^n a_i^2/i$$

under constraints $\sum_{i=1}^n a_i = 1$ and $a_i > 0, i = 1, ..., n$. Using the Lagrange-multiplier or the inequality

$$\left(\sum_{i=1}^n a_i^2/i\right)\left(\sum_{i=1}^n i\right) \geq (\sum_{i=1}^n a_i)^2 = 1,$$

we can obtain that $a_i = i/\sum_{i=1}^n i = 2i/(n(n+1))$. That is,

$$\bar{Y}_n^* = \frac{\sum_{i=1}^n iY_i}{n(n+1)/2}.$$

(b) Note

$$(\bar{Y}_n^* - \mu) = \frac{\sum_{i=1}^n i(Y_i-\mu)}{n(n+1)/2} = \frac{\sum_{i=1}^n \sqrt{i}\epsilon_i}{n(n+1)/2}.$$

Apply the Lindeberg CLT, we obtain

$$\sqrt{\frac{n(n+1)}{2}}(\bar{Y}_n^* - \mu) \to_d N(0, \sigma^2).$$

To verify the Lindeberg condition, note that the total variance $\sigma_n^2 = \sigma^2$ and for any $\delta > 0$,

$$\sum_{i=1}^n E\left[\frac{i}{n(n+1)/2}\epsilon_i^2 I(\sqrt{i}|\epsilon_i| > \delta\sigma_n\sqrt{\frac{n(n+1)}{2}})\right] \leq E[\epsilon_1^2 I(\epsilon_1 > \sqrt{\frac{n+1}{2}}\sigma_n)] \to 0.$$

3. (a) The information matrix for μ and σ is

$$I(\theta) = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 2/\sigma^2 \end{pmatrix}.$$

Since $\dot{q}_\theta = (-\sigma/\mu^2, 1/\mu)$ and $\dot{l}_\theta = ((Y-\mu)/\sigma^2, -1/\sigma + (Y-\mu)^2/\sigma^3)$, we obtain that the efficient influence function for CV is

$$\dot{q}_\theta I(\theta)^{-1}\dot{l}_\theta = \frac{(Y-\mu)^2 - \sigma^2}{2\mu\sigma} - \frac{\sigma(Y-\mu)}{\mu^2}$$

and that the information bound is

$$\dot{q}_\theta I(\theta)^{-1}\dot{q}_\theta' = \frac{\sigma^4}{\mu^4} + \frac{\sigma^2}{2\mu^2}.$$

(b) Clearly,

$$\widehat{CV} = g(\sum_{i=1}^n (Y_i - \bar{Y}_n)^2/n, \bar{Y}_n),$$

where $g(x,y) = \sqrt{x}/y$. Since

$$\sqrt{n}\left\{(\sum_{i=1}^n (Y_i - \bar{Y}_n)^2/n, \bar{Y}_n) - (\sigma^2, \mu)\right\} \to_d N\left(0, \begin{pmatrix} 2\sigma^4 & 0 \\ 0 & \sigma^2 \end{pmatrix}\right),$$

the Delta method gives

$$\sqrt{n}(\widehat{CV} - \sigma/\mu) \to_d N(0, \sigma^4/\mu^4 + \sigma^2/(2\mu^2)).$$

(c) We need to maximize the log-likelihood function

$$-n\log\sqrt{2\pi\sigma^2} - \frac{1}{2\sigma^2}\sum_{i=1}^n (Y_i - \mu)^2$$

in the region $\mu \geq 1$ and $\sigma > 0$. For fixed σ, this is a concave quadratic function with mode at $\bar{Y}_n$; thus, it is easy to see that $\hat{\mu}_n = \bar{Y}_n I(\bar{Y}_n \geq 1) + I(\bar{Y}_n < 1)$. Then $\hat{\sigma}_n = \sqrt{\sum_{i=1}^n (Y_i - \hat{\mu}_n)^2/n}$. Note

$$\sqrt{n}(\bar{Y}_n - \hat{\mu}_n) = \sqrt{n}(\bar{Y}_n - 1)I(\bar{Y}_n < 1)$$

and for $\epsilon > 0$,

$$P(|\sqrt{n}(\bar{Y}_n - 1)I(\bar{Y}_n < 1)| \geq \epsilon) \leq P(\bar{Y}_n < 1) \to_p 0$$

from the assumption that the true mean is larger than 1. We then conclude that $\sqrt{n}(\sum_{i=1}^n Y_i^2/n, \bar{Y}_n)$ should have the same asymptotic distribution as $\sqrt{n}(\sum_{i=1}^n Y_i^2/n, \hat{\mu}_n)$. Therefore, $\hat{\sigma}_n/\hat{\mu}_n$ has the same asymptotic distribution as $\widehat{CV}$. One may also sought the MLE theory to obtain the distribution.

4. (a) The observed likelihood function is

$$\prod_{i=1}^{n} \{\phi(Y_i - \mu)\}^{R_i} \{\Phi(y_0 - \mu)\}^{1-R_i},$$

where ϕ and Φ are respectively the density and CDF of the standard normal distribution.

(b) For $n = 1$, if μ_1 and μ_2 give the same likelihood function, consider $R = 1$, i.e., $Y > y_0$, we obtain

$$\phi(Y - \mu_1) = \phi(Y - \mu_2).$$

This should be true for any $Y > y_0$. It is easy to conclude $\mu_1 = \mu_2$.

(c) The likelihood equation for μ is

$$\sum_{i=1}^{n} \left[(Y_i - \mu)R_i - (1 - R_i)\frac{\phi(y_0 - \mu)}{\Phi(y_0 - \mu)} \right] = 0.$$

The Newton-Raphson iteration is

$$\mu^{(k+1)} = \mu^{(k)} - \left\{ -\sum_{i=1}^{n} R_i + \sum_{i=1}^{n}(1 - R_i) \left[\frac{\phi'(y_0 - \mu^{(k)})}{\Phi(y_0 - \mu^{(k)})} - \frac{\phi(y_0 - \mu^{(k)})^2}{\Phi(y_0 - \mu^{(k)})^2} \right] \right\}^{-1} \times \left\{ \sum_{i=1}^{n} \left[(Y_i - \mu^{(k)})R_i - (1 - R_i)\frac{\phi(y_0 - \mu^{(k)})}{\Phi(y_0 - \mu^{(k)})} \right] \right\}.$$

(d) The complete score equation for μ is

$$\sum_{i=1}^{n}(Y_i - \mu) = 0.$$

Thus, in the M-step, we have

$$\mu^{(k+1)} = \frac{1}{n}\sum_{i=1}^{n} E[Y_i | R_1 Y_1, ..., R_n Y_n, R_1, ..., R_n; \mu^{(k)}] = \frac{1}{n}\left\{ \sum_{i=1}^{n}(R_i Y_i + (1 - R_i)E[Y_i | Y_i > y_0; \mu^{(k)}]) \right\}.$$

Clearly, the E-step evaluates $E[Y_i | Y_i > y_0; \mu^{(k)}]$ using

$$\frac{\int_{y_0}^{\infty} y\phi(y - \mu^{(k)})dy}{1 - \Phi(y_0 - \mu^{(k)})} = \mu^{(k)} + \frac{\phi(y_0 - \mu^{(k)})}{1 - \Phi(y_0 - \mu^{(k)})}.$$

(e) By the MLE theory, $\sqrt{n}(\hat{\mu}_n - \mu) \to_d N(0, I(\mu)^{-1})$, where

$$\begin{aligned} I(\mu) &= E\left[R - (1 - R)\left\{ \frac{\phi'(y_0 - \mu)}{\Phi(y_0 - \mu)} - \frac{\phi(y_0 - \mu)^2}{\Phi(y_0 - \mu)^2} \right\} \right] \\ &= 1 - \Phi(y_0) - \phi'(y_0 - \mu) + \frac{\phi(y_0 - \mu)^2}{\Phi(y_0 - \mu)}. \end{aligned}$$

(f) The confidence interval can be constructed as

$$(\hat{\mu}_n - \Phi^{-1}(0.975)\sqrt{I(\hat{\mu}_n)^{-1}}, \hat{\mu}_n + \Phi^{-1}(0.975)\sqrt{I(\hat{\mu}_n)^{-1}}).$$