Solution to Final Exam

1. (a) The joint density is given as

\[
\prod_{i=1}^{n} \theta X_i^{\theta - 1} I(0 < X_i < 1) = \prod_{i=1}^{n} \frac{I(0 < X_i < 1)}{X_i} \exp\left\{ \theta \sum_{i=1}^{n} \log X_i + n \log \theta \right\}.
\]

This is a one-parameter exponential family with the canonical form. The complete statistic for \( \theta \) is \( \sum_{i=1}^{n} \log X_i \).

(b) After differentiating the log-likelihood function with respect to \( \theta \), we obtain the score equation

\[
0 = \frac{n}{\theta} + \sum_{i=1}^{n} \log X_i.
\]

Thus the MLE is \( \hat{\theta}_n = -n/ \sum_{i=1}^{n} \log X_i \). The information bound for \( \theta \) is \( -\mathbb{E}\left[-1/\theta^2\right]^{-1} = \theta^2 \). By the MLE theory,

\[
\sqrt{n}(\hat{\theta}_n - \theta) \rightarrow_d N(0, \theta^2).
\]

(c) By the CLT,

\[
\sqrt{n}(\bar{X}_n - E[X]) \rightarrow_d N(0, \text{Var}(X)).
\]

Since \( E[X] = \int_{0}^{1} \theta x^\theta dx = \theta/(\theta + 1) \) and \( \text{Var}(X) = \int_{0}^{1} \theta x^{\theta+1} dx - E[X]^2 = \theta/((\theta + 1)^2(\theta + 2)) \),

\[
\sqrt{n}(\bar{X}_n - \frac{\theta}{\theta + 1}) \rightarrow_d N(0, \frac{\theta}{(\theta + 1)^2(\theta + 2)}).
\]

By the Delta method with \( g(x) = x/(1 - x) \), we obtain

\[
\sqrt{n}(\hat{\delta}_n - g(\frac{\theta}{\theta + 1})) \rightarrow_d g'(x)\bigg|_{x=\theta/(\theta+1)} N(0, \frac{\theta}{(\theta + 1)^2(\theta + 2)}).
\]

That is,

\[
\sqrt{n}(\hat{\delta}_n - \theta) \rightarrow_d N(0, \frac{\theta(\theta + 1)^2}{\theta + 2}).
\]

(d) The ARE is equal to \( \theta(\theta + 2)/(\theta + 1)^2 \).

2. (a) It is clear that the joint distribution is an exponential family with the complete sufficient statistic \( T_n = \sum_{i=1}^{n} X_i \). Since \( P(X_1 = 0) = e^{-\lambda} \). The UMVU estimator for \( e^{-\lambda} \) is equal to

\[
\hat{\lambda}_n = P(X_1 = 0 | T_n = t) = \frac{P(X_1 = 0)P(X_2 + \ldots + X_n = t)}{P(X_1 + \ldots + X_n = t)}
= \frac{e^{-\lambda}((n-1)\lambda)^{t}e^{-(n-1)\lambda}/t!}{(n\lambda)^{t}e^{-n\lambda}/t!} = (1 - \frac{1}{n})^t.
\]
(b) Since $T_n \sim \text{Poisson}(n\lambda)$,

$$\text{Var}(\hat{\delta}_n) = \sum_{k=0}^{\infty} \left(1 - \frac{1}{n}\right)^{2k} \frac{(n\lambda)^k e^{-n\lambda}}{k!} - e^{-2\lambda} = e^{-2\lambda}(e^{\lambda/n} - 1)$$

The information bound for estimating $e^{-\lambda}$ is given by $e^{-2\lambda}I^{-1}_\lambda/n$, where $I_\lambda$ is the information matrix for $\lambda$. Since $I_\lambda = 1/\lambda$, the information bound for estimating $e^{-\lambda}$ is $e^{-2\lambda}\lambda/n$. The variance of the UMVE estimator $\hat{\delta}_n$ does not achieve the Cramér-Rao bound from the inequality $e^{\lambda/n} - 1 > \lambda/n$.

(c) First, by the Taylor expansion,

$$\sqrt{n}(T_n \log(1 - \frac{1}{n}) + \lambda) = \sqrt{n}(T_n(-\frac{1}{n} + \frac{a_n}{n^2}) + \lambda) = \sqrt{n}(-\frac{T_n}{n} + \lambda) + \frac{\sqrt{n}a_n T_n}{2n^2},$$

where $a_n$ is a constant between 0 and 2. Since $T_n/n \to_a \lambda$,

$$\frac{\sqrt{n}a_n T_n}{n^2} = \frac{a_n T_n}{\sqrt{n}} \to_p 0.$$ 

On the other hand, $\sqrt{n}(-T_n/n + \lambda) \to_d N(0, \lambda)$. From the Slutsky’s theorem, we obtain

$$\sqrt{n}(T_n \log(1 - \frac{1}{n}) + \lambda) \to_d N(0, \lambda).$$

By the Delta method with $g(x) = \exp\{x\}$, we obtain

$$\sqrt{n}(\hat{\delta}_n - e^{-\lambda}) = \sqrt{n}(g(T_n \log(1 - \frac{1}{n})) - g(-\lambda)) \to_d N(0, e^{-2\lambda}).$$

3. (a) The likelihood function for $(\alpha, \beta)$ is

$$\frac{1}{\sqrt{2\pi}} \exp\{-\frac{(Y - \alpha - \beta X)^2}{2}\} f(X).$$

The score functions for $\alpha$ and $\beta$ are

$$i_\alpha = Y - \alpha - \beta X,$$

$$i_\beta = X(Y - \alpha - \beta X).$$

The information matrix for $(\alpha, \beta)$ is equal to

$$\left(\begin{array}{cc}
1 & E[X] \\
E[X] & E[X^2]
\end{array}\right).$$

Thus, the efficient score function for $\beta$ is

$$i_\beta - I_{\beta\alpha} I^{-1}_{\alpha\alpha} i_\alpha = (X - E[X])(Y - \alpha - \beta X).$$

The efficiency bound for estimating $\beta$ is $I^{-1}_{\beta\beta}\alpha = \text{Var}(X)^{-1}$. 

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(b) The sufficient and necessary condition is \( I_{\alpha\beta} = 0 \), i.e., \( E[X] = 0 \).

4. (a) Fix \( M > 0 \), when \( n \) is large enough, \( P(|X_n| > a_n) \leq P(|X_n| \geq M) \). From the Portmanteau theorem, since the set outside of \((-M, M)\) is close set, \( \limsup_n P(|X_n| \geq M) \leq P(|X| \geq M) \). We obtain

\[
\limsup_n P(|X_n| > a_n) \leq P(|X| \geq M).
\]

Since \( M \) is arbitrary, \( P(|X_n| > a_n) \to 0 \). The result holds from the fact \( P(I(|X_n| > a_n) \geq \epsilon) \leq P(|X_n| > a_n) \).

(b) From (a), \( I(|X_n| \leq a_n) \to_p 1 \). From the Slutsky’s theorem, \( X_n I(|X_n| \leq a_n) \to_d X \).