Solution to Final Exam

1. (a) The joint density is given as

$$\prod_{i=1}^{n} \theta X_i^{\theta-1} I(0 < X_i < 1) = \prod_{i=1}^{n} \frac{I(0 < X_i < 1)}{X_i} \exp\{\theta \sum_{i=1}^{n} \log X_i + n \log \theta\}.$$

This is a one-parameter exponential family with the canonical form. The complete statistic for θ is $\sum_{i=1}^{n} \log X_i$.

(b) After differentiating the log-likelihood function with respect to θ, we obtain the score equation

$$0 = \frac{n}{\theta} + \sum_{i=1}^{n} \log X_i.$$

Thus the MLE is $\hat{\theta}_n = -n/\sum_{i=1}^{n} \log X_i$. The information bound for θ is $-E[-1/\theta^2]^{-1} = \theta^2$. By the MLE theory,

$$\sqrt{n}(\hat{\theta}_n - \theta) \to_d N(0, \theta^2).$$

(c) By the CLT,

$$\sqrt{n}(\bar{X}_n - E[X]) \to_d N(0, Var(X)).$$

Since $E[X] = \int_0^1 \theta x^\theta dx = \theta/(\theta+1)$ and $Var(X) = \int_0^1 \theta x^{\theta+1} dx - E[X]^2 = \theta/((\theta+1)^2(\theta+2))$,

$$\sqrt{n}(\bar{X}_n - \frac{\theta}{\theta+1}) \to_d N(0, \frac{\theta}{(\theta+1)^2(\theta+2)}).$$

By the Delta method with $g(x) = x/(1-x)$, we obtain

$$\sqrt{n}(\hat{\delta}_n - g(\frac{\theta}{\theta+1})) \to_d g'(x)\Big|_{x=\theta/(\theta+1)} N(0, \frac{\theta}{(\theta+1)^2(\theta+2)}).$$

That is,

$$\sqrt{n}(\hat{\delta}_n - \theta) \to_d N(0, \frac{\theta(\theta+1)^2}{\theta+2}).$$

(d) The ARE is equal to $\theta(\theta+2)/(\theta+1)^2$.

2. (a) It is clear that the joint distribution is an exponential family with the complete sufficient statistic $T_n = \sum_{i=1}^{n} X_i$. Since $P(X_1 = 0) = e^{-\lambda}$. The UMVU estimator for $e^{-\lambda}$ is equal to

$$\hat{\delta}_n = P(X_1 = 0 | T_n = t) = \frac{P(X_1 = 0) P(X_2 + ... + X_n = t)}{P(X_1 + ... + X_n = t)}$$
$$= \frac{e^{-\lambda}((n-1)\lambda)^t e^{-(n-1)t}/t!}{(n\lambda)^t e^{-n\lambda}/t!} = (1 - \frac{1}{n})^t.$$

(b) Since $T_n \sim Poisson(n\lambda)$,

$$Var(\hat{\delta}_n) = \sum_{k=0}^{\infty} (1 - \frac{1}{n})^{2k} \frac{(n\lambda)^k e^{-n\lambda}}{k!} - e^{-2\lambda} = e^{-2\lambda}(e^{\lambda/n} - 1)$$

The information bound for estimating $e^{-\lambda}$ is given by $e^{-2\lambda} I_\lambda^{-1}/n$, where I_λ is the information matrix for λ. Since $I_\lambda = 1/\lambda$, the information bound for estimating $e^{-\lambda}$ is $e^{-2\lambda}\lambda/n$. The variance of the UMVE estimator $\hat{\delta}_n$ does not achieve the Cramér-Rao bound from the inequality $e^{\lambda/n} - 1 > \lambda/n$.

(c) First, by the Taylor expansion,

$$\sqrt{n}(T_n \log(1 - \frac{1}{n}) + \lambda) = \sqrt{n}(T_n(-\frac{1}{n} + \frac{a_n}{n^2}) + \lambda) = \sqrt{n}(-\frac{T_n}{n} + \lambda) + \frac{\sqrt{n} a_n T_n}{2n^2},$$

where a_n is a constant between 0 and 2. Since $T_n/n \to_{a.s.} \lambda$,

$$\frac{\sqrt{n} a_n T_n}{n^2} = \frac{a_n}{\sqrt{n}} \frac{T_n}{n} \to_p 0.$$

On the other hand, $\sqrt{n}(-T_n/n + \lambda) \to_d N(0, \lambda)$. From the Slutsky's theorem, we obtain

$$\sqrt{n}(T_n \log(1 - \frac{1}{n}) + \lambda) \to_d N(0, \lambda).$$

By the Delta method with $g(x) = \exp\{x\}$, we obtain

$$\sqrt{n}(\hat{\delta}_n - e^{-\lambda}) = \sqrt{n}(g(T_n \log(1 - \frac{1}{n})) - g(-\lambda)) \to_d N(0, e^{-2\lambda}\lambda).$$

3. (a) The likelihood function for (α, β) is

$$\frac{1}{\sqrt{2\pi}} \exp\{-\frac{(Y - \alpha - \beta X)^2}{2}\} f(X).$$

The score functions for α and β are

$$\dot{l}_\alpha = Y - \alpha - \beta X,$$
$$\dot{l}_\beta = X(Y - \alpha - \beta X).$$

The information matrix for (α, β) is equal to

$$\begin{pmatrix} 1 & E[X] \\ E[X] & E[X^2] \end{pmatrix}.$$

Thus, the efficient score function for β is

$$\dot{l}_\beta - I_{\beta\alpha} I_{\alpha\alpha}^{-1} \dot{l}_\alpha = (X - E[X])(Y - \alpha - \beta X).$$

The efficiency bound for estimating β is $I_{\beta\beta\cdot\alpha}^{-1} = Var(X)^{-1}$.

(b) The sufficient and necessary condition is $I_{\alpha\beta} = 0$, i.e., $E[X] = 0$.

4. (a) Fix $M > 0$, when n is large enough, $P(|X_n| > a_n) \leq P(|X_n| \geq M)$. From the Portmanteau theorem, since the set outside of $(-M, M)$ is close set, $\limsup_n P(|X_n| \geq M) \leq P(|X| \geq M)$. We obtain

$$\limsup_n P(|X_n| > a_n) \leq P(|X| \geq M).$$

Since M is arbitrary, $P(|X_n| > a_n) \to 0$. The result holds from the fact $P(I(|X_n| > a_n) > \epsilon) \leq P(|X_n| > a_n)$.

(b) From (a), $I(|X_n| \leq a_n) \to_p 1$. From the Slutsky's theorem, $X_n I(|X_n| \leq a_n) \to_d X$.