BIOS 760 Final Exam (Solution)

1. (a) It is MCAR.

   (b) The joint likelihood function is

$$\prod_{i=1}^{n}\left\{(2\pi)^{-1}\exp\{-\frac{R_i(Y_i-\beta X_i)^2}{2}-\frac{X_i^2}{2}\}p_i^R(1-p)^{1-R_i}\right\}.$$

   (c) It is an exponential family with a complete and sufficient statistic

$$T_n=\sum_{i=1}^{n}R_iY_iX_i.$$

   (d) Since $E[T_n]=np\sum_{i=1}^n X_i^2\beta$, the UMVUE for β is $T/(np\sum_{i=1}^n X_i^2)$.

   (e) The information for β is

$$\sum_{i=1}^{n}E\left\{R_iX_i^2|X_i\right\}=p\sum_{i=1}^{n}X_i^2.$$

   So the CR-lower bound is is $1/(p\sum_{i=1}^n X_i^2)$. Since $Var(T_n/(np\sum_{i=1}^n X_i^2)|X_1,..,X_n)=1/(p\sum_{i=1}^n X_i^2)$, the UMVUE attains this lower bound.

2. (a) It is MAR.

   (b) When β is the truth,

$$E[R_i/p(W_i)(Y_i-\beta X_i)X_i]=E[E[R_i/p(W_i)(Y_i-\beta X_i)X_i]|Y_i,X_i,W_i]]=E[(Y_i-\beta X_i)X_i]=0.$$

   (c) The estimator is

$$\tilde{\beta}=\frac{\sum_{i=1}^n R_iY_iX_i/p(W_i)}{\sum_{i=1}^n R_iX_i^2/p(W_i)}.$$

   Since

$$\sqrt{n}\left\{\begin{pmatrix}n^{-1}\sum_{i=1}^n R_iY_iX_i/p(W_i)\\ n^{-1}\sum_{i=1}^n R_iX_i^2/p(W_i)\end{pmatrix}-\begin{pmatrix}\beta\\1\end{pmatrix}\right\}$$

$$\to_d N\left\{0,\begin{pmatrix}Var(R_iY_iX_i/p(W_i)) & cov(R_iY_iX_i/p(W_i),R_iX_i^2/p(W_i))\\ cov(R_iY_iX_i/p(W_i),R_iX_i^2/p(W_i)) & Var(R_iX_i^2/p(W_i))\end{pmatrix}\right\},$$

   by the Delta method,

$$\sqrt{n}(\tilde{\beta}-\beta)\to_d$$

$$N\left(0,(1,-\beta)\begin{pmatrix}Var(R_iY_iX_i/p(W_i)) & cov(R_iY_iX_i/p(W_i),R_iX_i^2/p(W_i))\\ cov(R_iY_iX_i/p(W_i),R_iX_i^2/p(W_i)) & Var(R_iX_i^2/p(W_i))\end{pmatrix}(1,-\beta)'\right).$$

3. (a) It is MNAR.

   (b) The joint likelihood is

$$\prod_{i=1}^{n}\left\{(2\pi)^{-1}\exp\left\{-\frac{(Y_i-\beta X_i)^2}{2}-\frac{X_i^2}{2}\right\}\right\}^{R_i}$$

$$\times\left\{\int_{y>c}(2\pi)^{-1}\exp\left\{-\frac{(y-\beta X_i)^2}{2}-\frac{X_i^2}{2}\right\}dy\right\}^{1-R_i}.$$

   (c) E-step: we compuete $\widehat{E}[Y_i]=Y_i$ if $R_i=1$ and

$$\widehat{E}[Y_i]=\frac{\int_c^\infty y\exp\left\{-\frac{(y-\beta X_i)^2}{2}\right\}dy}{\int_c^\infty \exp\left\{-\frac{(y-\beta X_i)^2}{2}\right\}dy}.$$

   M-step: we update β as

$$\frac{\sum_{i=1}^n \widehat{E}[Y_i]X_i}{\sum_{i=1}^n X_i^2}.$$