BIOS 760 Final, 2014: Solution

1. (a) The likelihood is $\beta^{2n}\left(\prod X_i\right)e^{-\beta\sum_{i=1}^n X_i}$ which implies that U_n is sufficient. It is also complete since this is a full rank exponential family. The density of a single observation is gamma with shape parameter $\alpha = 2$ and scale parameter β which implies that its n-fold convolution is gamma with shape parameter $2n$ and scale parameter β, which implies U_n has density $u \mapsto \beta^{2n}u^{2n-1}e^{-\beta u}/\Gamma(2n)$.

   (b) The UMVUE (if it exists) is that function of U_n which is unbiased for β. Note that

$$EU_n^{-t} = \int_0^\infty \frac{\beta^{2n-t}u^{2n-t}e^{-\beta u}}{\Gamma(2n-t)}du \cdot \frac{\beta^t\Gamma(2n-t)}{\Gamma(2n)},$$

   which implies that $EU_n^{-1} = \beta/(2n-1)$ and

$$EU_n^{-2} = \frac{\beta^2}{(2n-1)(2n-2)}.$$

   Thus the variance of U_n^{-1} is

$$\beta^2\left(\frac{1}{(2n-1)(2n-2)} - \frac{1}{(2n-1)^2}\right) = \frac{\beta^2}{(2n-1)^2(2n-2)},$$

   and thus

$$\tilde{\beta}_n = \frac{2n-1}{U_n} \text{ and } \tilde{M}_n = \frac{\beta^2}{2n-2}.$$

   (c) The log-likelihood is proportional to $2n\log\beta - \beta U_n$, and thus $\hat{\beta}_n = 2n/U_n$. Since $E\hat{\beta}_n = 2n\beta/(2n-1)$ and the variance of $\hat{\beta}_n$ is

$$\frac{(2n)^2\beta^2}{(2n-1)^2(2n-2)},$$

   the MSE is

$$\hat{M}_n = \frac{\beta^2}{(2n-1)^2} + \frac{(2n)^2\beta^2}{(2n-1)^2(2n-2)} = \frac{(4n^2+2n-2)\beta^2}{(2n-1)^2(2n-2)}.$$

   (d) From (b) and (c) above, $\hat{M}_n - \tilde{M}_n =$

$$\frac{\beta^2}{2n-2}\left(\frac{4n^2+2n-2-4n^2+4n-1}{(2n-1)^2}\right) = \frac{(6n-3)\beta^2}{(2n-1)^2(2n-2)} = \frac{3\beta^2}{(2n-1)(2n-2)}.$$

   (e) From (b) and (c) above, when $n = 1$, both $\tilde{M}_n$ and $\hat{M}_n$ are infinite.

2. (a) This is a mixture of two gamma densities, and thus $EX = (1/2)(1/\beta)+(1/2)(2/\beta) = 3/(2\beta)$ and $EX^2 = (1/2)(2/\beta^2) + (1/2)(6/\beta^2) = 4/(\beta^2)$. Thus

$$\mathrm{var}(X) = \frac{4}{\beta^2} - \frac{9}{4\beta^2} = \frac{7}{4\beta^2},$$

and hence

$$\sqrt{n}\left(\bar{X}_n - \frac{3}{2\beta}\right) \to_d N\left(0, \frac{7}{4\beta^2}\right).$$

Applying the delta method to the function $x \mapsto g(x) = 3/(2x)$, where $\dot{g}(\mu) = -3/(2\mu^2)$, and $\mu = 3/(2\beta)$, we obtain that $\dot{g}(\mu) = 2\beta/3$; and thus

$$\sqrt{n}(\tilde{\beta}_n - \beta_0) \to_d N\left(0, \frac{7[\dot{g}(\mu)]^2}{4\beta^2}\right) = N\left(0, \frac{7\beta^2}{9}\right).$$

(b) $\ell_\beta(X) = \log\left((1/2)\beta e^{-\beta X}(1+\beta X)\right) = -\log 2 + \log\beta - \beta X + \log(1+\beta X)$. Hence

$$\dot{\ell}_\beta(X) = \frac{1}{\beta} - X + \frac{X}{1+\beta X},$$

and thus

$$-\ddot{\ell}_\beta(X) = \frac{1}{\beta^2} + \frac{X^2}{(1+\beta X)^2}.$$

Hence the given form for I_0 is obtained.

(c) i. Since

$$\begin{aligned}\frac{\partial}{\partial\beta}\ddot{\ell}_\beta(X) &= \frac{2}{\beta^3} + \frac{2X^3}{(1+\beta X)^3} \\ &= \frac{2}{\beta^3}\left(1 + \frac{\beta^3 X^3}{(1+\beta X)^3}\right) \\ &\le \frac{4}{\beta^3},\end{aligned}$$

the mean value theorem yields that for some $\tilde{\beta}_n^*$ in between β_n^* and β_0,

$$|I_n(\beta_n^*) - I_n(\beta_0)| \le \frac{4}{(\tilde{\beta}_n^*)^3}|\beta_n^* - \beta_0| \le \frac{4}{(\beta_n^* \wedge \beta_0)^3}|\beta_n^* - \beta_0|.$$

Thus $|I_n(\beta_n^*) - I_n(\beta_0)| = o_P(1)$ which implies that $I_n(\beta_n^*) \to_P I_0$ since $I_n(\beta_0) \to_{a.s} I_0$ by the strong law of large numbers.

ii. By Taylor expansion,

$$\begin{aligned}\hat{\beta}_n - \beta_0 &= \tilde{\beta}_n - \beta_0 + \frac{n^{-1}\sum_{i=1}^n \dot{\ell}_{\beta_0}(X_i) + \ddot{\ell}_{\beta_n^*}(X_i)(\tilde{\beta}_n - \beta_0)}{I_n(\tilde{\beta}_n)} \\ &= \tilde{\beta}_n - \beta_0 + [I_n(\tilde{\beta}_n)]^{-1} n^{-1}\sum_{i=1}^n \dot{\ell}_{\beta_0}(X_i) - \frac{I_n(\beta_n^*)}{I_n(\tilde{\beta}_n)}(\tilde{\beta}_n - \beta_0),\end{aligned}$$

where β_n^* is between $\tilde{\beta}_n$ and β_0. Now, by part (i) above,

$$\begin{aligned}\sqrt{n}(\hat{\beta}_n - \beta_0) &= \sqrt{n}(\tilde{\beta}_n - \beta_0) - (1 + o_P(1))\sqrt{n}(\tilde{\beta}_n - \beta_0) \\ &\quad + (I_0^{-1} + o_P(1))n^{-1/2}\sum_{i=1}^{n} \dot{\ell}_{\beta_0}(X_i) \\ &= n^{-1/2}\sum_{i=1}^{n} I_0^{-1}\dot{\ell}_{\beta_0}(X_i) + o_P(1),\end{aligned}$$

which means that $\hat{\beta}_n$ is asymptotically linear with influence function equal to the efficient influence function, and thus it is both regular and asymptotically efficient.