BIOS 760 Final, 2014

1. Let $X_1, \ldots, X_n$ be i.i.d. with density $\beta^2 x e^{-\beta x}$ on the positive reals, where $0 < \beta < \infty$, and let $U_n = \sum_{i=1}^n X_i$. Do the following:

   (a) (3 points) Show that U_n is a sufficient and complete statistic for estimating β, and derive the density of U_n.

   (b) (3 points) Compute the UMVUE of β, which we will denote $\tilde{\beta}_n$, and give the mean squared error (MSE) of $\tilde{\beta}_n$, and let this MSE be denoted $\tilde{M}_n$.

   (c) (3 points) Compute the MLE of β, which we will denote $\hat{\beta}_n$, and give the MSE of $\hat{\beta}_n$, and let this MSE be denoted $\hat{M}_n$.

   (d) (3 points) Show that
   $$\hat{M}_n - \tilde{M}_n = \frac{3\beta^2}{(2n-1)(2n-2)}.$$

   (e) (2 bonus points) What happens to $\tilde{M}_n$ and $\hat{M}_n$ when $n = 1$?

2. Let $X_1, \ldots, X_n$ be i.i.d. with density $\beta e^{-\beta x}/2 + \beta^2 x e^{-\beta x}/2$ on the positive reals, where $0 < \beta < \infty$. Let $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$, and do the following:

   (a) (3 points) Let $\tilde{\beta}_n = 3/(2\bar{X}_n)$ and show that $\sqrt{n}(\tilde{\beta}_n - \beta_0)$ is asymptotically normal with mean zero and variance $\sigma^2(\beta_0)$, and give the form of $\sigma^2(\beta_0)$, where β_0 is the true value of β.

   (b) (2 points) Derive the score function $\dot{\ell}_\beta(X_i)$ for a single observation and show that the Fisher Information for this model is
   $$I_0 = \frac{1}{\beta_0^2}\left[1 + E\left(\frac{\beta_0^2 X^2}{(1+\beta_0 X)^2}\right)\right].$$

   (c) Let
   $$I_n(\beta) = \frac{1}{\beta^2}\left[1 + n^{-1}\sum_{i=1}^n \left(\frac{\beta^2 X_i^2}{(1+\beta X_i)^2}\right)\right],$$
   and show that
   $$\hat{\beta}_n = \tilde{\beta}_n + \frac{n^{-1}\sum_{i=1}^n \dot{\ell}_{\tilde{\beta}_n}(X_i)}{I_n(\tilde{\beta}_n)}$$
   is an asymptotically regular and efficient estimator of β_0 using the following steps (over):

i. (3 bonus points) Show that for any β sufficiently close to β_0,

$$|I_n(\beta) - I_n(\beta_0)| \leq \frac{4|\beta - \beta_0|}{(\beta \wedge \beta_0)^3},$$

where $\wedge$ denotes the minimum, and verify that $I_n(\beta_n^*)$ converges to I_0 in probability whenever β_n^* converges to β_0 in probability.

ii. (3 points) Show that

$$\sqrt{n}(\hat{\beta}_n - \beta_0) = n^{-1/2} \sum_{i=1}^{n} I_0^{-1} \dot{\ell}_{\beta_0}(X_i) + o_P(1),$$

and state why this completes the proof.