Solution to BIOS760: 2011 FALL SEMESTER FINAL EXAM

1. (a) The joint density is given as

$$(2\theta)^{-n} I(|X_1| \le \theta, ..., |X_n| \le \theta) = (2\theta)^{-n} I(|X|_{(n)} \le \theta).$$

Thus, $|X|_{(n)}$ is sufficient. Suppose $E[Q(|X|_{(n)})] = 0$. Since

$$P(|X|_n \le x) = x^n/\theta^n I(0 \le x \le \theta) + I(x \ge \theta),$$

we have

$$E[Q(|X|_{(n)})] = \int_0^\theta Q(x) n x^{n-1}/\theta^n dx = 0.$$

That is, $\int_0^\theta Q(x) x^{(n-1)} dx = 0$ so after taking derivatives, we obtain $Q(\theta) = 0$. This implies that $|X|_{(n)}$ is also complete.

(b) Note that if $x \le 0$, $P(n(1 - |X|_{(n)}/\theta) \le x) = 0$; if $x > 0$, for n large enough,

$$P(n(1 - |X|_{(n)}/\theta) \le x) = P(|X|_{(n)} \ge \theta - \theta x/n) = 1 - (1 - x/n)^n \to 1 - e^{-x}.$$

We conclude that $n(1 - |X|_{(n)}/\theta)$ converges in distribution to the exponential distribution.

(c) Clearly, the MLE for θ is $|X|_{(n)}$. Thus, the MLE for $g(\theta)$ is $g(|X|_{(n)})$. By the delta method, we obtain

$$n(g(\theta) - g(|X|_{(n)}))/\theta \to_d g'(\theta) Exp(1).$$

Thus, $n(g(\theta) - g(|X|_{(n)})) \to_d \theta g'(\theta) Exp(1)$. That is, the limiting distribution is the exponential distribution with mean $\theta g'(\theta)$. Let $t_{1-\alpha}$ be the $(1-\alpha)$th quantile of $Exp(1)$, i.e., $t_{1-\alpha} = -\log(\alpha)$. Then by estimating θ using $|X|_{(n)}$, a confidence interval for $g(\theta)$ is $[0, g(|X|_{(n)}) + |X|_{(n)} g'(|X|_{(n)}) t_{1-\alpha}/n]$.

(d) To find the UMVUE, we need to determine $Q(|X|_{(n)})$ satisfying $E[Q(|X|_{(n)})] = g(\theta)$. Equivalently,

$$\int_0^\theta Q(x) n x^{n-1} dx = g(\theta)\theta^n.$$

Taking derivatives on both sides, we obtain

$$Q(x) = g(x) + g'(x) x/n.$$

The UMVUE for θ is $T_n = g(|X|_{(n)}) + g'(|X|_{(n)})|X|_{(n)}/n$.

(e) We note

$$n(g(\theta) - T_n) = n(g(\theta) - g(|X|_{(n)})) - g'(|X|_{(n)})|X|_{(n)}.$$

Since the previous result implies that $|X|_{(n)} \to_p \theta$, we obtain from the Slutsky lemma that

$$n(g(\theta) - T_n) \to_d \theta g'(\theta) Exp(1) - g'(\theta)\theta.$$

A confidence interval for $g(\theta)$ is $[0, T_n + |X|_{(n)} g'(|X|_{(n)}) t_{1-\alpha}/n - g'(|X|_{(n)})|X|_{(n)}]$.

2. (a) The conditional density is given as

$$\int_\xi e^{\beta X}\xi\exp\{-Ye^{\beta X}\xi\}\Gamma(\lambda)^{-1}\lambda^\lambda\xi^{\lambda-1}\exp\{-\lambda\xi\}d\xi = e^{\beta X}(1+Ye^{\beta X}/\lambda)^{-(\lambda+1)}.$$

Thus, the likelihood function is given as

$$\prod_{i=1}^n \left[e^{\beta X_i}(1+Y_ie^{\beta X_i}/\lambda)^{-(\lambda+1)}f(X_i)\right],$$

where $f(x)$ is the density of X.

(b) Assume that X has a full rank with positive probability. Then β is identifiable since

$$\left[e^{\beta X_i}(1+Y_ie^{\beta X_i}/\lambda)^{-(\lambda+1)}f(X_i)\right] = \left[e^{\beta^* X_i}(1+Y_ie^{\beta^* X_i}/\lambda)^{-(\lambda+1)}f^*(X_i)\right]$$

leads to $\beta X_i = \beta^* X_i$.

(c) It follows from

$$E[Y|X] = [E[Y|X,\xi]|X] = E[e^{-\beta X}\xi^{-1}|X] = e^{-\beta X}c,$$

where $c = E[\xi^{-1}] = \lambda/(\lambda-1)$. From the results for estimating equation, if $\tilde{\beta}$ is the solution, then $\sqrt{n}(\tilde{\beta}-\beta)$ converges in distribution to a normal distribution with mean zero and variance (sandwich formula)

$$E[cX^2e^{-\beta X}]^{-1}E[X^2(Y-ce^{-\beta X})^2]E[cX^2e^{-\beta X}]^{-1}.$$

(d) The score equation is

$$\sum_{i=1}^n \left[X_i - \frac{(\lambda+1)Y_iX_ie^{\beta X_i}}{\lambda+Y_ie^{\beta X_i}}\right] = 0.$$

The Newton-Raphson iteration is

$$\beta^{(k+1)} = \beta^{(k)} - \left\{-\sum_{i=1}^n \frac{(\lambda+1)Y_iX_i^2e^{\beta^{(k)}X_i}}{\lambda+Y_ie^{\beta^{(k)}X_i}} + \sum_{i=1}^n \frac{(\lambda+1)Y_i^2X_i^2e^{2\beta^{(k)}X_i}}{(\lambda+Y_ie^{\beta^{(k)}X_i})^2}\right\}^{-1}$$

$$\times\sum_{i=1}^n \left[X_i - \frac{(\lambda+1)Y_iX_ie^{\beta^{(k)}X_i}}{\lambda+Y_ie^{\beta^{(k)}X_i}}\right] = 0.$$

(e) The complete log-likelihood function (concerning β) is

$$\prod_{i=1}^n \left[e^{\beta X_i}\xi_i\exp\{-Y_ie^{\beta X_i}\xi_i\}\right].$$

Then the M-step solves equation

$$\sum_{i=1}^n \left[X_i - Y_ie^{\beta X_i}X_iw_i\right] = 0.$$

In the E-step, we calculate w_i as the conditional expectation of ξ_i given observed data: since $\xi|Y_i,X_i$ follows a gamma distribution with parameter $(1+\lambda,[Y_ie^{\beta X_i}+\lambda]^{-1})$,

$$w_i = E[\xi_i|Y_i,X_i] = (1+\lambda)/[Y_ie^{\beta X_i}+\lambda].$$