BIOS 760: Final 2010

For the following problems, assume that $T_1, \ldots, T_n$ are i.i.d. random variables and that each T_i, $i = 1, \ldots, n$ follows an exponential distribution with rate λ, i.e., the density is $\lambda^{-1}e^{-t/\lambda}$, $t > 0$. Our goal is to estimate λ.

As a reminder, here are a few facts regarding exponential random variable T_i with rate λ.

- $E[T_i] = \lambda$ and $\text{Var}(T_i) = \lambda^2$.
- $\sum_{i=1}^n T_i \sim \Gamma(n, \lambda)$ where the density of $\Gamma(n, \lambda)$ is given by $\lambda^{-n}t^{n-1}e^{-t/\lambda}/(n-1)!$
- The memoryless property:
  $P(T_i > s + t | T_i > t) = P(T_i > s)$ for all $t, s \geq 0$

1. Answer the following questions about UMVUE estimation of λ:

   (a) (3 points) Find the UMVUE estimator for λ. Does the estimator attain the information bound?

   (b) (3 points) Use the fact that $T_{(1)} = \min\{T_1, \ldots, T_n\}$ has exponential distribution with rate λ/n to derive another unbiased estimator for λ based on $T_{(1)}$. Compare to the UMVUE estimator derived in Problem 1(a). Which one is better? Give a qualitative explanation.

   (c) (3 points) Find the UMVUE estimator for λ^k for any positive integer k.

2. Assume that $C_1, \ldots, C_n$ are i.i.d. random censoring times with distribution function G and density g. Assume also that $C_1, \ldots, C_n$ are independent of the failure times T_i, where T_i are exponential with rate λ as defined above. Assume that we observe i.i.d. pairs $(Z_1, \Delta_1), \ldots, (Z_n, \Delta_n)$, where $Z_i = \min\{T_i, C_i\}$ and $\Delta_i = 1_{\{T_i \leq C_i\}}$. Do the following:

   (a) (2 points) Show that the observed likelihood function of λ based on the observed pairs $(Z_1, \Delta_1), \ldots, (Z_n, \Delta_n)$ has the following form:

   $$\prod_{1=1}^{n} \left(\lambda^{-1}e^{-Z_i/\lambda}(1 - G(Z_i))\right)^{\Delta_i} \left(e^{-Z_i/\lambda}g(Z_i)\right)^{1-\Delta_i}.$$

   (b) (2 points) Show that the maximum likelihood estimator for $\hat{\lambda}$ based on the above likelihood has the form:

   $$\hat{\lambda} = \frac{\sum_{i=1}^n Z_i}{\sum_{i=1}^n \Delta_i}$$

(c) (3 points) Calculate the information bound for λ in this censored setting.

Hint: It is easier to change the parametrization from λ to $\theta = 1/\lambda$. Use the likelihood in question 2a and substitute θ accordingly. Compute the information bound for θ. Then, calculate the bound for λ.

(d) (3 bonus points) Derive the asymptotic distribution of $\hat{\lambda}$.

3. Maximum likelihood estimation for the above censored model can also be accomplished using the EM algorithm. Do the following:

(a) (3 points) Show that the conditional density of T_i given λ, $Z_i = z_i$ and $\Delta_i = d_i$ has the form:

$$f(t_i|z_i, d_i, \lambda) = d_i 1_{\{t_i = z_i\}} + (1 - d_i) 1_{\{t_i > z_i\}} \lambda^{-1} e^{-(t_i - z_i)/\lambda}.$$

(b) (2 bonus points) Show that the log likelihood for (T_i, Z_i, Δ_i) given $T_i = t_i$, $Z_i = z_i$, and $\Delta_i = d_i$, which we denote $\log f(t_i, z_i, d_i|\lambda)$, is given by

$$\log f(t_i, z_i, d_i|\lambda) = 1_{\{t_i \geq z_i\}} \left(-\log \lambda - \frac{t_i}{\lambda} \right) + d_i 1_{\{t_i = z_i\}} \log(1 - G(z_i)) + (1 - d_i) 1_{\{t_i > z_i\}} \log g(z_i) \, .$$

Hint: You may use the fact that $f(t_i, z_i, d_i|\lambda) = f(z_i, d_i|\lambda) f(t_i|z_i, d_i, \lambda)$, and then show that this product equals

$$\left(\lambda^{-1} e^{-t_i/\lambda} \right)^{1_{\{t_i \geq z_i\}}} (1 - G(z_i))^{d_i 1_{\{t_i = z_i\}}} g(z_i)^{(1-d_i) 1_{\{t_i > z_i\}}}.$$

(c) (3 points) Use the previous questions to derive the E-step of an EM algorithm for estimating λ.

Hint 1: Recall that given an estimator $\lambda^{(k)}$, the E-step consists of computing

$$E \left[\sum_{i=1}^{n} \log f(T_i, Z_i, \Delta_i|\lambda) \middle| \{Z_1 = z_1, \Delta_1 = d_1\}, \ldots, \{Z_n = z_n, \Delta_n = d_n\}, \lambda^{(k)} \right].$$

Hint 2: Note that this is done by computing

$$\sum_{i=1}^{n} \int_0^\infty \log f(t_i, z_i, d_i|\lambda) f(t_i|z_i, d_i, \lambda^{(k)}) dt_i$$

(d) (3 points) Derive the M-step of an EM algorithm for estimating λ. Explain how to estimate λ using the E and M steps.

(e) (3 bonus points) Find a fixed point of the algorithm. In other words, find a $\tilde{\lambda}$ such that $\tilde{\lambda}^{(k+1)} = \tilde{\lambda}^{(k)} = \tilde{\lambda}$ for all k. Explain why $\tilde{\lambda}$ is the maximum point of the likelihood. Compare to the estimator $\hat{\lambda}$ of question 2b.