BIOS260 FINAL EXAM

December 13, 2005

1. (15pts) Let $X_n \rightarrow_{a.s.} X$, $X_n \rightarrow_p X$ and $X_n \rightarrow_d X$ denote three modes of convergence for the sequence of random variables $\{X_n\}$.

   (a) (5pts) Give the order from the strongest convergence to the weakest one. That is, the first one implies the second one and the second one implies the third one.

   (b) (5pts) Give one example when the second one does not imply the first one.

   (c) (5pts) Give one example when the third one does not imply the second one.

2. (15pts) Suppose that $X_1, ..., X_n$ are independent random variables and X_i has density

$$\lambda\omega_i \exp\{-\lambda\omega_i x\} I(x > 0),$$

   where ω_i's are fixed positive constants. λ is the only parameter.

   (a) (4pts) Find a complete sufficient statistic for $1/\lambda$ and derive the UMVUE of $1/\lambda$.

   (b) (4pts) Does the variance of the UMVUE attain the Cramér-Rao lower bound? Justify your answer.

   (c) (3pts) What is the maximum likelihood estimator of $1/\lambda$?

   (d) (4pts) Derive the asymptotic distribution of the maximum likelihood estimator.

3. (20pts) A total of n i.i.d AIDS patients are selected from a population. We let X_i be the true CD4+ cell count of the ith patient. Assume $X_i \sim N(\mu, 1)$. For the ith patient, clinician measures his/her cell count twice and record as (Y_{i1}, Y_{i2}). To account for the error in the measurements, we assume that given X_i,

$$Y_{i1} = X_i + \epsilon_{i1}, \quad Y_{i2} = X_i + \epsilon_{i2},$$

   where ϵ_{i1} and ϵ_{i2} are independent and follow $N(0, \sigma^2)$. Thus, we only observe n pairs of i.i.d data $(Y_{i1}, Y_{i2}), i = 1, ..., n$. Both μ and σ^2 are parameters.

   (a) (3pts) What is the joint distribution of (Y_{i1}, Y_{i2})? Write out the explicit expression the observed log-likelihood function.

   (b) (4pts) What is the efficient information bound and the efficient influence function for μ?

   (c) (3pts) Treating $(Y_{i1}, Y_{i2}, X_i), i = 1, ..., n$ as the complete data and $X_i, i = 1, ..., n$ as the missing data, write out the complete score equations for μ and σ^2.

   (d) (4pts) Derive the conditional distribution of X_i given (Y_{i1}, Y_{i2}).

   (e) (6pts) Derive the EM algorithm for calculating the maximum likelihood estimates of μ and σ^2. You need to give the explicit expressions in both E-step and M-step.