BIOS260 Final Exam

December 09, 8:00am-11:00am

1. Suppose $X_1, ..., X_n$ are i.i.d. with common density function $\theta x^{\theta-1} I(0 < x < 1)$ where $\theta > 0$.

   (a) (3 points) Show that the distributions for the data form a one-parameter exponential family and identify a complete sufficient statistic.

   (b) (5 points) Find the maximum likelihood estimator $\hat{\theta}_n$ and determine the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$ as $n \to \infty$.

   (c) (5 points) Determine the asymptotic distribution of $\sqrt{n}(\hat{\delta}_n - \theta)$ where $\hat{\delta}_n = \bar{X}_n/(1 - \bar{X}_n)$ and $\bar{X}_n = \sum_{i=1}^n X_i/n$.

   (d) (2 points) Compute the asymptotic relative efficiency of the estimator $\hat{\delta}_n$ with respect to the maximum likelihood estimator $\hat{\theta}_n$.

2. Suppose $X_1, ..., X_n$ are i.i.d. from $Poisson(\lambda)$. Consider estimation of $e^{-\lambda}$.

   (a) (5 points) Determine the UMVU estimator for $e^{-\lambda}$, denoted by $\hat{\delta}_n$. You can use the fact that the summation of K independent $Poisson(\lambda)$ is $Poisson(K\lambda)$.

   (b) (5 points) Show that the variance of $\hat{\delta}_n$ does not achieve the Cramér-Rao bound.

   (c) (5 points) Using the expansion $\log(1 - x) = -x + ax^2$ for $0 < x < 1/2$ where $a \in (0, 2)$, derive the asymptotic distribution of $\sqrt{n}(\hat{\delta}_n - e^{-\lambda})$.

3. Suppose that X has a known density $f(x)$ and that given X, $Y \sim N(\alpha + \beta X, 1)$.

   (a) (5 points) Calculate the efficient score function and the efficiency bound for β.

   (b) (5 points) Under what condition is the information bound for β unchanged, whether α is known or unknown?

4. Let X_n be a sequence of random variables such that

$$X_n \to_d X,$$

   where X is a finite random variable. For any positive constant sequence $a_n \to \infty$, prove

   (a) (5 points) $I(|X_n| > a_n) \to_p 0$.

   (b) (5 points) $X_n I(|X_n| \leq a_n) \to_d X$.