CHAPTER 3: LARGE SAMPLE THEORY

Introduction

- Why large sample theory
  - studying small sample property is usually difficult and complicated
  - large sample theory studies the limit behavior of a sequence of random variables, say X_n.
  - example: $\bar{X}_n \to \mu$, $\sqrt{n}(\bar{X}_n - \mu)$

Modes of Convergence

- Convergence almost surely

Definition 3.1 X_n is said to *converge almost surely* to X, denoted by $X_n \rightarrow_{a.s.} X$, if there exists a set $A \subset \Omega$ such that $P(A^c) = 0$ and for each $\omega \in A$, $X_n(\omega) \rightarrow X(\omega)$ in real space.

- Equivalent condition

$$\{\omega : X_n(\omega) \to X(\omega)\}^c$$

$$= \cup_{\epsilon>0} \cap_n \{\omega : \sup_{m\geq n} |X_m(\omega) - X(\omega)| > \epsilon\}$$

$\Rightarrow$ $X_n \to_{a.s.} X$ iff

$$P(\sup_{m\geq n} |X_m - X| > \epsilon) \to 0$$

- Convergence in probability

Definition 3.2 X_n is said to *converge in probability* to X, denoted by $X_n \to_p X$, if for every $\epsilon > 0$,

$$P(|X_n - X| > \epsilon) \to 0.$$

- Convergence in moments/means

Definition 3.3 X_n is said to *converge in rth mean* to X, denote by $X_n \rightarrow_r X$, if

$E[|X_n - X|^r] \rightarrow 0$ as $n \rightarrow \infty$ for functions $X_n, X \in L_r(P)$,

where $X \in L_r(P)$ means $\int |X|^r dP < \infty$.

- Convergence in distribution

Definition 3.4 X_n is said to *converge in distribution* of X, denoted by $X_n \to_d X$ or $F_n \to_d F$ (or $L(X_n) \to L(X)$ with L referring to the "law" or "distribution"), if the distribution functions F_n and F of X_n and X satisfy

$F_n(x) \to F(x)$ as $n \to \infty$ for each continuity point x of F.

- Uniform integrability

Definition 3.5 A sequence of random variables $\{X_n\}$ is *uniformly integrable* if

$$\lim_{\lambda \to \infty} \limsup_{n \to \infty} E\left\{|X_n| I(|X_n| \geq \lambda)\right\} = 0.$$

- A note
  - Convergence almost surely and convergence in probability are the same as we defined in measure theory.
  - Two new definitions are
    - convergence in rth mean
    - convergence in distribution

- "convergence in distribution"
  - is very different from others
  - example: a sequence $X, Y, X, Y, X, Y, ...$ where X and Y are $N(0, 1)$; the sequence converges in distribution to $N(0, 1)$ but the other modes do not hold.
  - "convergence in distribution" is important for asymptotic statistical inference.

- Relationship among different modes

Theorem 3.1 A. If $X_n \rightarrow_{a.s.} X$, then $X_n \rightarrow_p X$.

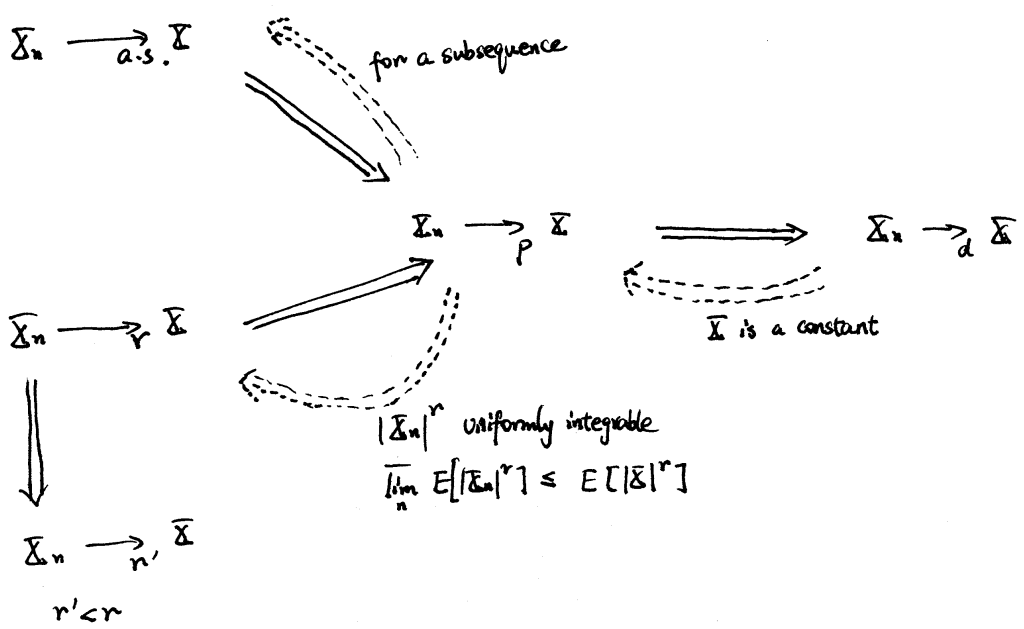
B. If $X_n \rightarrow_p X$, then $X_{n_k} \rightarrow_{a.s.} X$ for some subsequence X_{n_k}.

C. If $X_n \rightarrow_r X$, then $X_n \rightarrow_p X$.

D. If $X_n \rightarrow_p X$ and $|X_n|^r$ is uniformly integrable, then $X_n \rightarrow_r X$.

E. If $X_n \rightarrow_p X$ and $\limsup_n E|X_n|^r \leq E|X|^r$, then $X_n \rightarrow_r X$.

F. If $X_n \rightarrow_r X$, then $X_n \rightarrow_{r'} X$ for any $0 < r' \leq r$.
G. If $X_n \rightarrow_p X$, then $X_n \rightarrow_d X$.
H. $X_n \rightarrow_p X$ if and only if for every subsequence $\{X_{n_k}\}$ there exists a further subsequence $\{X_{n_{k,l}}\}$ such that $X_{n_{k,l}} \rightarrow_{a.s.} X$.
I. If $X_n \rightarrow_d c$ for a constant c, then $X_n \rightarrow_p c$.



$X_n \to_{a.s.} X$
for a subsequence
$X_n \to_p X$
$X_n \to_d X$
X is a constant
$X_n \to_r X$
$|X_n|^r$ uniformly integrable
$\overline{\lim}_n E[|X_n|^r] \leq E[|X|^r]$
$X_n \to_{r'} X$
$r' < r$

Proof

A and B follow from the results in the measure theory.

Prove C. *Markov inequality*: for any increasing function $g(\cdot)$ and random variable Y, $P(|Y| > \epsilon) \leq E[\frac{g(|Y|)}{g(\epsilon)}]$.

$\Rightarrow P(|X_n - X| > \epsilon) \leq E[\frac{|X_n - X|^r}{\epsilon^r}] \to 0.$

Prove D. It is sufficient to show that for any subsequence of $\{X_n\}$, there exists a further subsequence $\{X_{n_k}\}$ such that $E|X_{n_k} - X|^r \to 0$.

For any subsequence of $\{X_n\}$, from B, there exists a further subsequence $\{X_{n_k}\}$ such that $X_{n_k} \to_{a.s.} X$. For any ϵ, there exists λ such that $\limsup_{n_k} E[|X_{n_k}|^r I(|X_{n_k}|^r \geq \lambda)] < \epsilon$.

Particularly, choose λ such that $P(|X|^r = \lambda) = 0$
$\Rightarrow$ $|X_{n_k}|^r I(|X_{n_k}|^r \geq \lambda) \to_{a.s.} |X|^r I(|X|^r \geq \lambda)$.

$\Rightarrow$ By the Fatou's Lemma,

$$E[|X|^r I(|X|^r \geq \lambda)] \leq \limsup_{n_k} E[|X_{n_k}|^r I(|X_{n_k}|^r \geq \lambda)] < \epsilon.$$

$\Rightarrow$

$$
\begin{aligned}
& E[|X_{n_k} - X|^r] \\
\leq\ & E[|X_{n_k} - X|^r I(|X_{n_k}|^r < 2\lambda, |X|^r < 2\lambda)] \\
& + E[|X_{n_k} - X|^r I(|X_{n_k}|^r \geq 2\lambda, \text{ or }, |X|^r \geq 2\lambda)] \\
\leq\ & E[|X_{n_k} - X|^r I(|X_{n_k}|^r < 2\lambda, |X|^r < 2\lambda)] \\
& + 2^r E[(|X_{n_k}|^r + |X|^r) I(|X_{n_k}|^r \geq 2\lambda, \text{ or }, |X|^r \geq 2\lambda)],
\end{aligned}
$$

where the last inequality follows from the inequality
$(x+y)^r \leq 2^r(\max(x,y))^r \leq 2^r(x^r + y^r), x \geq 0, y \geq 0.$

When n_k is large, the second term is bounded by

$$2 * 2^r \{E[|X_{n_k}|^r I(|X_{n_k}| \geq \lambda)] + E[|X|^r I(|X| \geq \lambda)]\} \leq 2^{r+1}\epsilon.$$

$\Rightarrow \limsup_n E[|X_{n_k} - X|^r] \leq 2^{r+1}\epsilon.$

Prove E. It is sufficient to show that for any subsequence of $\{X_n\}$, there exists a further subsequence $\{X_{n_k}\}$ such that $E[|X_{n_k} - X|^r] \to 0$.

For any subsequence of $\{X_n\}$, there exists a further subsequence $\{X_{n_k}\}$ such that $X_{n_k} \to_{a.s.} X$. Define

$$Y_{n_k} = 2^r(|X_{n_k}|^r + |X|^r) - |X_{n_k} - X|^r \geq 0.$$

$\Rightarrow$ By the Fatou's Lemma,

$$\int \liminf_{n_k} Y_{n_k} dP \leq \liminf_{n_k} \int Y_{n_k} dP.$$

It is equivalent to

$$2^{r+1} E[|X|^r] \leq \liminf_{n_k} \left\{2^r E[|X_{n_k}|^r] + 2^r E[|X|^r] - E[|X_{n_k} - X|^r]\right\}.$$

Prove F. The Hölder inequality:

$$\int |f(x)g(x)|d\mu \leq \left\{\int |f(x)|^p d\mu(x)\right\}^{1/p} \left\{\int |g(x)|^p d\mu(x)\right\}^{1/q},$$

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Choose $\mu = P$, $f = |X_n - X|^{r'}, g \equiv 1$ and $p = r/r', q = r/(r - r')$ in the Hölder inequality

$\Rightarrow$

$$E[|X_n - X|^{r'}] \leq E[|X_n - X|^r]^{r'/r} \to 0.$$

Prove G. $X_n \to_p X$. If $P(X = x) = 0$, then for any $\epsilon > 0$,

$$
\begin{aligned}
& P(|I(X_n \le x) - I(X \le x)| > \epsilon) \\
= \quad & P(|I(X_n \le x) - I(X \le x)| > \epsilon, |X - x| > \delta) \\
& + P(|I(X_n \le x) - I(X \le x)| > \epsilon, |X - x| \le \delta) \\
\le \quad & P(X_n \le x, X > x + \delta) + P(X_n > x, X < x - \delta) \\
& + P(|X - x| \le \delta) \\
\le \quad & P(|X_n - X| > \delta) + P(|X - x| \le \delta).
\end{aligned}
$$

The first term converges to zero since $X_n \to_p X$.

The second term can be arbitrarily small if δ is small, since $\lim_{\delta \to 0} P(|X - x| \le \delta) = P(X = x) = 0$.

$\Rightarrow I(X_n \le x) \to_p I(X \le x)$

$\Rightarrow F_n(x) = E[I(X_n \le x)] \to E[I(X \le x)] = F(x)$.

Prove H. One direction follows from B.

To prove the other direction, use the contradiction. Suppose there exists $\epsilon > 0$ such that $P(|X_n - X| > \epsilon)$ does not converge to zero. $\Rightarrow$ find a subsequence $\{X_{n'}\}$ such hat $P(|X_{n'} - X| > \epsilon) > \delta$ for some $\delta > 0$.

However, by the condition, there exists a further subsequence $X_{n''}$ such that $X_{n''} \to_{a.s.} X$ then $X_{n''} \to_p X$ from A. Contradiction!

Prove I. Let $X \equiv c$.

$$P(|X_n - c| > \epsilon) \leq 1 - F_n(c+\epsilon) + F_n(c-\epsilon)$$

$$\rightarrow 1 - F_X(c+\epsilon) + F(c-\epsilon) = 0.$$

- Some counter-examples

(*Example 1*) Suppose that X_n is degenerate at a point $1/n$; i.e., $P(X_n = 1/n) = 1$. Then X_n converges in distribution to zero. Indeed, X_n converges almost surely.

(*Example 2*) $X_1, X_2, ...$ are i.i.d with standard normal distribution. Then $X_n \to_d X_1$ but X_n does not converge in probability to X_1.

(*Example 3*) Let Z be a random variable with a uniform distribution in $[0, 1]$. Let
$X_n = I(m2^{-k} \leq Z < (m+1)2^{-k})$ when $n = 2^k + m$
where $0 \leq m < 2^k$. Then it is shown that X_n converges in probability to zero but not almost surely. This example is already given in the second chapter.

(*Example 4*) Let Z be $Uniform(0,1)$ and let $X_n = 2^n I(0 \leq Z < 1/n)$. Then $E[|X_n|^r]] \to \infty$ but X_n converges to zero almost surely.

- Result for convergence in rth mean

Theorem 3.2 (Vitali's theorem) Suppose that $X_n \in L_r(P)$, i.e., $\|X_n\|_r < \infty$, where $0 < r < \infty$ and $X_n \to_p X$. Then the following are equivalent:

A. $\{|X_n|^r\}$ are uniformly integrable.

B. $X_n \to_r X$.

C. $E[|X_n|^r] \to E[|X|^r]$.

- One sufficient condition for uniform integrability
  - *Liapunov condition*: there exists a positive constant ϵ_0 such that $\limsup_n E[|X_n|^{r+\epsilon_0}] < \infty$

$$E[|X_n|^r I(|X_n|^r \geq \lambda)] \leq \frac{E[|X_n|^{r+\epsilon_0}|]}{\lambda^{\epsilon_0}}$$

Integral inequalities

- *Young's inequality*

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}, \quad a, b > 0,$$

where the equality holds if and only if $a = b$.

$\log x$ is concave:

$$\log(\frac{1}{p}|a|^p + \frac{1}{q}|b|^q) \geq \frac{1}{p}\log|a|^p + \frac{1}{q}\log|b|.$$

Geometric interpretation (insert figure here):

- *Hölder inequality*

$$\int |f(x)g(x)|d\mu(x) \leq \left\{\int |f(x)|^p d\mu(x)\right\}^{\frac{1}{p}} \left\{\int |g(x)|^q d\mu(x)\right\}^{\frac{1}{q}}.$$

  - in the Young's inequality, let
    $a = f(x)/\left\{\int |f(x)|^p d\mu(x)\right\}^{1/p}$
    $b = g(x)/\left\{\int |g(x)|^q d\mu(x)\right\}^{1/q}$.
  - when $\mu = P$ and $f = X(\omega)$, $g = 1$, $\mu_r^{s-t}\mu_t^{r-s} \geq \mu_s^{r-t}$ where $\mu_r = E[|X|^r]$ and $r \geq s \geq t \geq 0$.
  - when $p = q = 2$, obtain *Cauchy-Schwartz inequality*:

$$\int |f(x)g(x)|d\mu(x) \leq \left\{\int f(x)^2 d\mu(x)\right\}^{\frac{1}{2}} \left\{\int g(x)^2 d\mu(x)\right\}^{\frac{1}{2}}.$$

- *Minkowski's inequality* $r > 1$,

$$\|X + Y\|_r \leq \|X\|_r + \|Y\|_r.$$

  - derivation:

$$E[|X + Y|^r] \leq E[(|X| + |Y|)|X + Y|^{r-1}]$$

$$\leq E[|X|^r]^{1/r}E[|X+Y|^r]^{1-1/r}+E[|Y|^r]^{1/r}E[|X+Y|^r]^{1-1/r}.$$

  - $\|\cdot\|_r$ in fact is a norm in the linear space $\{X : \|X\|_r < \infty\}$. Such a normed space is denoted as $L_r(P)$.

- *Markov's inequality*

$$P(|X| \geq \epsilon) \leq \frac{E[g(|X|)]}{g(\epsilon)},$$

where $g \geq 0$ is a increasing function in $[0, \infty)$.

- Derivation:

$$P(|X| \geq \epsilon) \leq P(g(|X|) \geq g(\epsilon))$$

$$= E[I(g(|X|) \geq g(\epsilon))] \leq E[\frac{g(|X|)}{g(\epsilon)}].$$

- When $g(x) = x^2$ and X replaced by $X - \mu$, obtain *Chebyshev's inequality*:

$$P(|X - \mu| \geq \epsilon) \leq \frac{Var(X)}{\epsilon^2}.$$

- Application of Vitali's theorem
  - $Y_1, Y_2, ...$ are i.i.d with mean μ and variance σ^2. Let $X_n = \bar{Y}_n$.
  - By the Chebyshev's inequality,

$$P(|X_n - \mu| > \epsilon) \leq \frac{Var(X_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0.$$

    $\Rightarrow$ $X_n \to_p \mu$.
  - From the Liapunov condition with $r = 1$ and $\epsilon_0 = 1$, $|X_n - \mu|$ satisfies the uniform integrability condition
    $\Rightarrow$

$$E[|X_n - \mu|] \to 0.$$

Convergence in Distribution

"Convergence in distribution is the most important mode of convergence in statistical inference."

Equivalent conditions

Theorem 3.3 (Portmanteau Theorem) The following conditions are equivalent.

(a). X_n converges in distribution to X.

(b). For any bounded continuous function $g(\cdot)$, $E[g(X_n)] \to E[g(X)]$.

(c). For any open set G in R, $\liminf_n P(X_n \in G) \geq P(X \in G)$.

(d). For any closed set F in R, $\limsup_n P(X_n \in F) \leq P(X \in F)$.

(e). For any Borel set O in R with $P(X \in \partial O) = 0$ where ∂O is the boundary of O, $P(X_n \in O) \to P(X \in O)$.

Proof

$(a) \Rightarrow (b)$. Without loss of generality, assume $|g(x)| \leq 1$. We choose $[-M, M]$ such that $P(|X| = M) = 0$.

Since g is continuous in $[-M, M]$, g is uniformly continuous in $[-M, M]$.
$\Rightarrow$ Partition $[-M, M]$ into finite intervals $I_1 \cup ... \cup I_m$ such that within each interval I_k, $\max_{I_k} g(x) - \min_{I_k} g(x) \leq \epsilon$ and X has no mass at all the endpoints of I_k (why?).

Therefore, if choose any point $x_k \in I_k, k = 1, ..., m$,

$$\begin{aligned}
& |E[g(X_n)] - E[g(X)]| \\
\leq\ & E[|g(X_n)|I(|X_n| > M)] + E[|g(X)|I(|X| > M)] \\
& +|E[g(X_n)I(|X_n| \leq M)] - \sum_{k=1}^{m} g(x_k)P(X_n \in I_k)| \\
& +|\sum_{k=1}^{m} g(x_k)P(X_n \in I_k) - \sum_{k=1}^{m} g(x_k)P(X \in I_k)| \\
& +|E[g(X)I(|X| \leq M)] - \sum_{k=1}^{m} g(x_k)P(X \in I_k)| \\
\leq\ & P(|X_n| > M) + P(|X| > M) \\
& +2\epsilon + \sum_{k=1}^{m} |P(X_n \in I_k) - P(X \in I_k)|.
\end{aligned}$$

$\Rightarrow \limsup_n |E[g(X_n)] - E[g(X)]| \leq 2P(|X| > M) + 2\epsilon$. Let $M \to \infty$ and $\epsilon \to 0$.

$(b)\Rightarrow(c)$. For any open set G, define $g(x) = 1 - \frac{\epsilon}{\epsilon + d(x, G^c)}$, where $d(x, G^c)$ is the minimal distance between x and G^c, $\inf_{y \in G^c} |x - y|$.

For any $y \in G^c$, $d(x_1, G^c) - |x_2 - y| \leq |x_1 - y| - |x_2 - y| \leq |x_1 - x_2|$,
$\Rightarrow$ $d(x_1, G^c) - d(x_2, G^c) \leq |x_1 - x_2|$.
$\Rightarrow$ $|g(x_1) - g(x_2)| \leq \epsilon^{-1} |d(x_1, G^c) - d(x_2, G^c)| \leq \epsilon^{-1} |x_1 - x_2|$.
$\Rightarrow$ $g(x)$ is continuous and bounded.
$\Rightarrow$ $E[g(X_n)] \to E[g(X)]$.

Note $0 \leq g(x) \leq I_G(x)$
$\Rightarrow$

$$\liminf_n P(X_n \in G) \geq \liminf_n E[g(X_n)] \to E[g(X)].$$

Let $\epsilon \to 0$ $\Rightarrow$ $E[g(X)]$ converges to $E[I(X \in G)] = P(X \in G)$.

$(c)\Rightarrow(d)$. This is clear by taking complement of F.

$(d) \Rightarrow (e)$. For any O with $P(X \in \partial O) = 0$,

$$\limsup_n P(X_n \in O) \leq \limsup_n P(X_n \in \bar{O}) \leq P(X \in \bar{O}) = P(X \in O),$$

$$\liminf_n P(X_n \in O) \geq \liminf_n P(X_n \in O^o) \geq P(X \in O^o) = P(X \in O).$$

$(e) \Rightarrow (a)$. Choose $O = (-\infty, x]$ with $P(X \in \partial O) = P(X = x) = 0$.

- Counter-examples
  - Let $g(x) = x$, a continuous but unbounded function. Let X_n be a random variable taking value n with probability $1/n$ and value 0 with probability $(1 - 1/n)$. Then $X_n \to_d 0$. However, $E[g(X)] = 1$ does not converge to 0.
  - The continuity at boundary in (e) is also necessary: let X_n be degenerate at $1/n$ and consider $O = \{x : x > 0\}$. Then $P(X_n \in O) = 1$ but $X_n \to_d 0$.

Weak Convergence and Characteristic Functions

Theorem 3.4 (Continuity Theorem) Let ϕ_n and ϕ denote the characteristic functions of X_n and X respectively. Then $X_n \rightarrow_d X$ is equivalent to $\phi_n(t) \rightarrow \phi(t)$ for each t.

Proof

To prove $\Rightarrow$ direction, from (b) in Theorem 3.1,

$$\phi_n(t) = E[e^{itX_n}] \to E[e^{itX}] = \phi(t).$$

The proof of $\Leftarrow$ direction consists of a few tricky constructions (skipped).

- One simple example $X_1, ..., X_n \sim Bernoulli(p)$

$$\phi_{\bar{X}_n}(t) = E[e^{it(X_1+...+X_n)/n}] = (1 = p + pe^{it/n})^n$$

$$= (1 - p + p + itp/n + o(1/n))^n \to e^{itp}.$$

Note the limit is the c.f. of $X = p$. Thus, $\bar{X}_n \to_d p$ so $\bar{X}_n$ converges in probability to p.

- Generalization to multivariate random vectors
  - $X_n \to_d X$ if and only if $E[\exp\{it'X_n\}] \to E[\exp\{it'X\}]$, where t is any k-dimensional constant
  - Equivalently, $t'X_n \to_d t'X$ for any t
  - to study the weak convergence of random vectors, we can reduce to study the weak convergence of one-dimensional linear combination of the random vectors
  - This is the well-known *Cramér-Wold's device*

Theorem 3.5 (The Cramér-Wold device) Random vector X_n in R^k satisfy $X_n \to_d X$ if and only $t'X_n \to_d t'X$ in R for all $t \in R^k$.

Properties of Weak Convergence

Theorem 3.6 (Continuous mapping theorem)
Suppose $X_n \to_{a.s.} X$, or $X_n \to_p X$, or $X_n \to_d X$. Then for any continuous function $g(\cdot)$, $g(X_n)$ converges to $g(X)$ almost surely, or in probability, or in distribution.

Proof

If $X_n \rightarrow_{a.s.} X, \Rightarrow g(X_n) \rightarrow_{a.s} g(X)$.

If $X_n \rightarrow_p X$, then for any subsequence, there exists a further subsequence $X_{n_k} \rightarrow_{a.s.} X$. Thus, $g(X_{n_k}) \rightarrow_{a.s.} g(X)$. Then $g(X_n) \rightarrow_p g(X)$ from (H) in Theorem 3.1.

To prove that $g(X_n) \rightarrow_d g(X)$ when $X_n \rightarrow_d X$, use (b) of Theorem 3.1.

- One remark

Theorem 3.6 concludes that $g(X_n) \to_d g(X)$ if $X_n \to_d X$ and g is continuous. In fact, this result still holds if $P(X \in C(g)) = 1$ where $C(g)$ contains all the continuity points of g. That is, if g's discontinuity points take zero probability of X, the continuous mapping theorem holds.

Theorem 3.7 (Slutsky theorem) Suppose $X_n \rightarrow_d X$, $Y_n \rightarrow_p y$ and $Z_n \rightarrow_p z$ for some constant y and z. Then $Z_n X_n + T_n \rightarrow_d zX + y$.

Proof

First show that $X_n + Y_n \to_d X + y$.

For any $\epsilon > 0$,

$$P(X_n + Y_n \leq x) \leq P(X_n + Y_n \leq x, |Y_n - y| \leq \epsilon) + P(|Y_n - y| > \epsilon)$$

$$\leq P(X_n \leq x - y + \epsilon) + P(|Y_n - y| > \epsilon).$$

$\Rightarrow \limsup_n F_{X_n+Y_n}(x) \leq \limsup_n F_{X_n}(x - y + \epsilon) \leq F_X(x - y + \epsilon).$

On the other hand,

$$P(X_n + Y_n > x) = P(X_n + Y_n > x, |Y_n - y| \leq \epsilon) + P(|Y_n - y| > \epsilon)$$

$$\leq P(X_n > x - y - \epsilon) + P(|Y_n - y| > \epsilon).$$

$\Rightarrow$

$$\limsup_n (1 - F_{X_n+Y_n}(x)) \leq \limsup_n P(X_n > x - y - \epsilon)$$

$$\leq \limsup_n P(X_n \geq x - y - 2\epsilon) \leq (1 - F_X(x - y - 2\epsilon)).$$

$\Rightarrow$ $F_X(x - y - 2\epsilon) \leq \liminf_n F_{X_n+Y_n}(x) \leq \limsup_n F_{X_n+Y_n}(x) \leq F_X(x + y + \epsilon)$.

$\Rightarrow$

$$F_{X+y}(x-) \leq \liminf_n F_{X_n+Y_n}(x) \leq \limsup_n F_{X_n+Y_n}(x) \leq F_{X+y}(x).$$

To complete the proof,

$$P(|(Z_n - z)X_n| > \epsilon) \leq P(|Z_n - z| > \epsilon^2) + P(|Z_n - z| \leq \epsilon^2, |X_n| > \frac{1}{\epsilon}).$$

$\Rightarrow$

$$\limsup_n P(|(Z_n - z)X_n| > \epsilon) \leq \limsup_n P(|Z_n - z| > \epsilon^2)$$

$$+ \limsup_n P(|X_n| \geq \frac{1}{2\epsilon}) \to P(|X| \geq \frac{1}{2\epsilon}).$$

$\Rightarrow$ that $(Z_n - z)X_n \to_p 0$.

Clearly $zX_n \to_d zX$ $\Rightarrow$ $Z_nX_n \to_d zX$ from the proof in the first half.

Again, using the first half's proof, $Z_nX_n + Y_n \to_d zX + y$.

- Examples
  - Suppose $X_n \to_d N(0,1)$. Then by continuous mapping theorem, $X_n^2 \to_d \chi_1^2$.
  - This example shows that g can be discontinuous in Theorem 3.6. Let $X_n \to_d X$ with $X \sim N(0,1)$ and $g(x) = 1/x$. Although $g(x)$ is discontinuous at origin, we can still show that $1/X_n \to_d 1/X$, the reciprocal of the normal distribution. This is because $P(X = 0) = 0$. However, in Example 3.6 where $g(x) = I(x > 0)$, it shows that Theorem 3.6 may not be true if $P(X \in C(g)) < 1$.

– The condition $Y_n \to_p y$, where y is a constant, is necessary. For example, let $X_n = X \sim Uniform(0,1)$. Let $Y_n = -X$ so $Y_n \to_d -\tilde{X}$, where $\tilde{X}$ is an independent random variable with the same distribution as X. However $X_n + Y_n = 0$ does not converge in distribution to $X - \tilde{X}$.

– Let $X_1, X_2, ...$ be a random sample from a normal distribution with mean μ and variance $\sigma^2 > 0$,

$$\sqrt{n}(\bar{X}_n - \mu) \to_d N(0, \sigma^2),$$

$$s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2 \to_{a.s} \sigma^2.$$

$\Rightarrow$

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{s_n} \to_d \frac{1}{\sigma}N(0, \sigma^2) \cong N(0, 1).$$

$\Rightarrow$ in large sample, t_{n-1} can be approximated by a standard normal distribution.

Representation of Weak Convergence

Theorem 3.8 (Skorohod's Representation Theorem) Let $\{X_n\}$ and X be random variables in a probability space $(\Omega, \mathcal{A}, P)$ and $X_n \rightarrow_d X$. Then there exists another probability space $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P})$ and a sequence of random variables $\tilde{X}_n$ and $\tilde{X}$ defined on this space such that $\tilde{X}_n$ and X_n have the same distributions, $\tilde{X}$ and X have the same distributions, and moreover, $\tilde{X}_n \rightarrow_{a.s.} \tilde{X}$.

- Quantile function

$$F^{-1}(p) = \inf\{x : F(x) \geq p\}.$$

Proposition 3.1 (a) F^{-1} is left-continuous.
(b) If X has continuous distribution function F, then $F(X) \sim Uniform(0,1)$.
(c) Let $\xi \sim Uniform(0,1)$ and let $X = F^{-1}(\xi)$. Then for all x, $\{X \leq x\} = \{\xi \leq F(x)\}$. Thus, X has distribution function F.

Proof

(a) Clearly, F^{-1} is nondecreasing. Suppose p_n increases to p then $F^{-1}(p_n)$ increases to some $y \le F^{-1}(p)$. Then $F(y) \ge p_n$ so $F(y) \ge p$. $\Rightarrow$ $F^{-1}(p) \le y$ $\Rightarrow$ $y = F^{-1}(p)$.

(b) $\{X \le x\} \subset \{F(X) \le F(x)\}$ $\Rightarrow$ $F(x) \le P(F(X) \le F(x))$.
$\{F(X) \le F(x) - \epsilon\} \subset \{X \le x\}$ $\Rightarrow$ $P(F(X) \le F(x) - \epsilon) \le F(x)$ $\Rightarrow$ $P(F(X) \le F(x)-) \le F(x)$.

Then if X is continuous, $P(F(X) \le F(x)) = F(x)$.

(c) $P(X \le x) = P(F^{-1}(\xi) \le x) = P(\xi \le F(x)) = F(x)$.

Proof

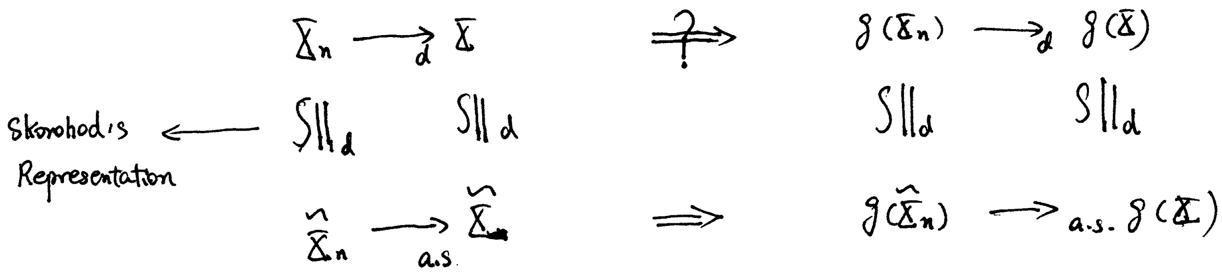
Let $(\tilde{\Omega}, \tilde{\mathcal{A}}, \tilde{P})$ be $([0,1], \mathcal{B} \cap [0,1], \lambda)$. Define $\tilde{X}_n = F_n^{-1}(\xi)$, $\tilde{X} = F^{-1}(\xi)$, where $\xi \sim Uniform(0,1)$. $\tilde{X}_n$ has a distribution F_n which is the same as X_n.

For any $t \in (0,1)$ such that there is at most one value x such that $F(x) = t$ (it is easy to see t is the continuous point of F^{-1}),
$\Rightarrow$ for any $z < x$, $F(z) < t$
$\Rightarrow$ when n is large, $F_n(z) < t$ so $F_n^{-1}(t) \geq z$.
$\Rightarrow$ $\liminf_n F_n^{-1}(t) \geq z$ $\Rightarrow$ $\liminf_n F_n^{-1}(t) \geq x = F^{-1}(t)$.

From $F(x+\epsilon) > t$, $F_n(x+\epsilon) > t$ so $F_n^{-1}(t) \leq x + \epsilon$.
$\Rightarrow$ $\limsup_n F_n^{-1}(t) \leq x + \epsilon$ $\Rightarrow$ $\limsup_n F_n^{-1}(t) \leq x$.

Thus $F_n^{-1}(t) \to F^{-1}(t)$ for almost every $t \in (0,1)$ $\Rightarrow$ $\tilde{X}_n \to_{a.s.} \tilde{X}$.

- Usefulness of representation theorem
  - For example, if $X_n \to_d X$ and one wishes to show some function of X_n, denote by $g(X_n)$, converges in distribution to $g(X)$:
  - see the diagram in Figure 2.


Skorohod's
Representation

- Alternative Proof for Slutsky Theorem

First, show $(X_n, Y_n) \to_d (X, y)$.

$$|\phi_{(X_n,Y_n)}(t_1, t_2) - \phi_{(X,y)}(t_1, t_2)| = |E[e^{it_1 X_n} e^{it_2 Y_n}] - E[e^{it_1 X} e^{it_2 y}]|$$

$$\leq |E[e^{it_1 X_n}(e^{it_2 Y_n} - e^{it_2 y})]| + |e^{it_2 y}||E[e^{it_1 X_n}] - E[e^{it_1 X}]|$$

$$\leq E[|e^{it_2 Y_n} - e^{it_2 y}|] + |E[e^{it_1 X_n}] - E[e^{it_1 X}]| \to 0.$$

Thus, $(Z_n, X_n) \to_d (z, X)$. Since $g(z, x) = zx$ is continuous,

$$\Rightarrow \quad Z_n X_n \to_d zX.$$

Since $(Z_n X_n, Y_n) \to_d (zX, y)$ and $g(x, y) = x + y$ is continuous,

$$\Rightarrow \quad Z_n X_n + Y_n \to_d zX + y.$$

Summation of Independent R.V.s

- Some preliminary lemmas

Proposition 3.2 (Borel-Cantelli Lemma) For any events A_n,

$$\sum_{i=1}^{\infty} P(A_n) < \infty$$

implies $P(A_n, i.o.) = P(\{A_n\}$ occurs infinitely often$) = 0$; or equivalently, $P(\cap_{n=1}^{\infty} \cup_{m\geq n} A_m) = 0$.

Proof

$$P(A_n, i.o) \leq P(\cup_{m\geq n} A_m) \leq \sum_{m\geq n} P(A_m) \to 0, \quad \text{as } n \to \infty.$$

- One result of the first Borel-cantelli lemma

If for a sequence of random variables, $\{Z_n\}$, and for any $\epsilon > 0$, $\sum_n P(|Z_n| > \epsilon) < \infty$, then $|Z_n| > \epsilon$ only occurs finite times.

$\Rightarrow Z_n \rightarrow_{a.s.} 0$.

Proposition 3.3 (Second Borel-Cantelli Lemma) For a sequence of independent events $A_1, A_2, ...,$ $\sum_{n=1}^{\infty} P(A_n) = \infty$ implies $P(A_n, i.o.) = 1$.

Proof Consider the complement of $\{A_n, i.o\}$.

$$P(\cup_{n=1}^{\infty} \cap_{m\geq n} A_m^c) = \lim_n P(\cap_{m\geq n} A_m^c) = \lim_n \prod_{m\geq n} (1 - P(A_m))$$

$$\leq \limsup_n \exp\{-\sum_{m\geq n} P(A_m)\} = 0.$$

- Equivalence lemma

Proposition 3.4 $X_1, ..., X_n$ are i.i.d with finite mean. Define $Y_n = X_n I(|X_n| \leq n)$. Then

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) < \infty.$$

Proof Since $E[|X_1|] < \infty$,

$$\sum_{n=1}^{\infty} P(|X| \geq n) = \sum_{n=1}^{\infty} nP(n \leq |X| < (n+1)) \leq \sum_{n=1}^{\infty} E[|X|] < \infty.$$

From the Borel-Cantelli Lemma, $P(X_n \neq Y_n, i.o) = 0$.

For almost every $\omega \in \Omega$, when n is large enough, $X_n(\omega) = Y_n(\omega)$.

Weak Law of Large Numbers

Theorem 3.9 (Weak Law of Large Number) If $X, X_1, ..., X_n$ are i.i.d with mean μ (so $E[|X|] < \infty$ and $\mu = E[X]$), then $\bar{X}_n \to_p \mu$.

Proof

Define $Y_n = X_n I(-n \le X_n \le n)$. Let $\bar{\mu}_n = \sum_{k=1}^n E[Y_k]/n$.

$P(|\bar{Y}_n - \bar{\mu}_n| \ge \epsilon) \le \frac{Var(\bar{Y}_n)}{\epsilon^2} \le \frac{\sum_{k=1}^n Var(X_k I(|X_k| \le k))}{n^2\epsilon^2}$.

$$
\begin{aligned}
& Var(X_k I(|X_k| \le k)) \le E[X_k^2 I(|X_k| \le k)] \\
= \quad & E[X_k^2 I(|X_k| \le k, |X_k| \ge \sqrt{k}\epsilon^2)] + E[X_k^2 I(|X_k| \le k, |X| \le \sqrt{k}\epsilon^2)] \\
\le \quad & kE[|X_k| I(|X_k| \ge \sqrt{k}\epsilon^2)] + k\epsilon^4,
\end{aligned}
$$

$\Rightarrow P(|\bar{Y}_n - \mu_n| \ge \epsilon) \le \frac{\sum_{k=1}^n E[|X| I(|X| \ge \sqrt{k}\epsilon^2)]}{n\epsilon^2} + \epsilon^2 \frac{n(n+1)}{2n^2}$. $\Rightarrow$ $\limsup_n P(|\bar{Y}_n - \mu_n| \ge \epsilon) \le \epsilon^2 \Rightarrow \bar{Y}_n - \bar{\mu}_n \to_p 0$.

$\bar{\mu}_n \to \mu \Rightarrow \bar{Y}_n \to_p \mu$. From Proposition 3.4 and subsequence arguments,

$\bar{X}_{nk} \to_{a.s.} \mu \Rightarrow X_n \to_p \mu$.

Strong Law of Large Numbers

Theorem 3.10 (Strong Law of Large Number) If $X_1, ..., X_n$ are i.i.d with mean μ then $\bar{X}_n \rightarrow_{a.s.} \mu$.

Proof

Without loss of generality, we assume $X_n \geq 0$ since if this is true, the result also holds for any X_n by $X_n = X_n^+ - X_n^-$.

Similar to Theorem 3.9, it is sufficient to show $\bar{Y}_n \to_{a.s.} \mu$, where $Y_n = X_n I(X_n \leq n)$.

Note $E[Y_n] = E[X_1 I(X_1 \leq n)] \to \mu$ so

$$\sum_{k=1}^{n} E[Y_k]/n \to \mu.$$

$\Rightarrow$ if we denote $\tilde{S}_n = \sum_{k=1}^n (Y_k - E[Y_k])$ and we can show $\tilde{S}_n/n \to_{a.s.} 0$, then the result holds.

$$Var(\tilde{S}_n) = \sum_{k=1}^{n} Var(Y_k) \leq \sum_{k=1}^{n} E[Y_k^2] \leq nE[X_1^2 I(X_1 \leq n)].$$

By the Chebyshev's inequality,

$$P(|\frac{\tilde{S}_n}{n}| > \epsilon) \leq \frac{1}{n^2\epsilon^2} Var(\tilde{S}_n) \leq \frac{E[X_1^2 I(X_1 \leq n)]}{n\epsilon^2}.$$

For any $\alpha > 1$, let $u_n = [\alpha^n]$.

$$\sum_{n=1}^{\infty} P(|\frac{\tilde{S}_{u_n}}{u_n}| > \epsilon) \leq \sum_{n=1}^{\infty} \frac{1}{u_n\epsilon^2} E[X_1^2 I(X_1 \leq u_n)] \leq \frac{1}{\epsilon^2} E[X_1^2 \sum_{u_n \geq X_1} \frac{1}{u_n}].$$

Since for any $x > 0$, $\sum_{u_n \geq x} \{\mu_n\}^{-1} < 2 \sum_{n \geq \log x / \log \alpha} \alpha^{-n} \leq Kx^{-1}$ for some constant K, $\Rightarrow$

$$\sum_{n=1}^{\infty} P(|\frac{\tilde{S}_{u_n}}{u_n}| > \epsilon) \leq \frac{K}{\epsilon^2} E[X_1] < \infty,$$

$\Rightarrow$ $\tilde{S}_{u_n}/u_n \rightarrow_{a.s.} 0$.

For any k, we can find $u_n < k \leq u_{n+1}$. Thus, since $X_1, X_2, ... \geq 0$,

$$\frac{\tilde{S}_{u_n}}{u_n}\frac{u_n}{u_{n+1}} \leq \frac{\tilde{S}_k}{k} \leq \frac{\tilde{S}_{u_{n+1}}}{u_{n+1}}\frac{u_{n+1}}{u_n}.$$

$\Rightarrow$

$$\mu/\alpha \leq \liminf_k \frac{\tilde{S}_k}{k} \leq \limsup_k \frac{\tilde{S}_k}{k} \leq \mu\alpha.$$

Since α is arbitrary number larger than 1, let $\alpha \to 1$ and we obtain $\lim_k \tilde{S}_k/k = \mu$.

Central Limit Theorems

- Preliminary result of c.f.

Proposition 3.5 Suppose $E[|X|^m] < \infty$ for some integer $m \geq 0$. Then

$$|\phi_X(t) - \sum_{k=0}^{m} \frac{(it)^k}{k!} E[X^k]|/|t|^m \to 0, \quad \text{as } t \to 0.$$

Proof

$$e^{itx} = \sum_{k=1}^{m} \frac{(itx)^k}{k!} + \frac{(itx)^m}{m!}[e^{it\theta x} - 1],$$

where $\theta \in [0, 1]$.

$\Rightarrow$

$$|\phi_X(t) - \sum_{k=0}^{m} \frac{(it)^k}{k!} E[X^k]|/|t|^m \leq E[|X|^m |e^{it\theta X} - 1|]/m! \to 0,$$

as $t \to 0$.

- Simple versions of CLT

Theorem 3.11 (Central Limit Theorem) If $X_1, ..., X_n$ are i.i.d with mean μ and variance σ^2 then $\sqrt{n}(\bar{X}_n - \mu) \rightarrow_d N(0, \sigma^2)$.

Proof

Denote $Y_n = \sqrt{n}(\bar{X}_n - \mu)$.

$$\phi_{Y_n}(t) = \left\{\phi_{X_1-\mu}(t/\sqrt{n})\right\}^n .$$

$\Rightarrow$ $\phi_{X_1-\mu}(t/\sqrt{n}) = 1 - \sigma^2 t^2/2n + o(1/n)$.

$\Rightarrow$

$$\phi_{Y_n}(t) \to \exp\{-\frac{\sigma^2 t^2}{2}\}.$$

Theorem 3.12 (Multivariate Central Limit Theorem) If $X_1, ..., X_n$ are i.i.d random vectors in R^k with mean μ and covariance $\Sigma = E[(X-\mu)(X-\mu)']$, then $\sqrt{n}(\bar{X}_n - \mu) \rightarrow_d N(0, \Sigma)$.

Proof

Use the Cramér-Wold's device.

- Liaponov CLT

Theorem 3.13 (Liapunov Central Limit Theorem)
Let $X_{n1}, ..., X_{nn}$ be independent random variables with $\mu_{ni} = E[X_{ni}]$ and $\sigma_{ni}^2 = Var(X_{ni})$. Let $\mu_n = \sum_{i=1}^n \mu_{ni}$, $\sigma_n^2 = \sum_{i=1}^n \sigma_{ni}^2$. If

$$\sum_{i=1}^n \frac{E[|X_{ni} - \mu_{ni}|^3]}{\sigma_n^3} \to 0,$$

then $\sum_{i=1}^n (X_{ni} - \mu_{ni})/\sigma_n \to_d N(0,1)$.

- Lindeberg-Feller CLT

Theorem 3.14 (Lindeberg-Fell Central Limit Theorem) Let $X_{n1}, ..., X_{nn}$ be independent random variables with $\mu_{ni} = E[X_{ni}]$ and $\sigma_{ni}^2 = Var(X_{ni})$. Let $\sigma_n^2 = \sum_{i=1}^n \sigma_{ni}^2$. Then both $\sum_{i=1}^n (X_{ni} - \mu_{ni})/\sigma_n \to_d N(0,1)$ and $\max\{\sigma_{ni}^2/\sigma_n^2 : 1 \leq i \leq n\} \to 0$ if and only if the Lindeberg condition

$$\frac{1}{\sigma_n^2}\sum_{i=1}^n E[|X_{ni}-\mu_{ni}|^2 I(|X_{ni}-\mu_{ni}| \geq \epsilon\sigma_n)] \to 0, \text{ for all } \epsilon > 0$$

holds.

- Proof of Liapunov CLT using Theorem 3.14

$$\frac{1}{\sigma_n^2}\sum_{i=1}^n E[|X_{nk}-\mu_{nk}|^2 I(|X_{nk}-\mu_{nk}|>\epsilon\sigma_n)]$$

$$\leq \frac{1}{\epsilon^3\sigma_n^3}\sum_{k=1}^n E[|X_{nk}-\mu_{nk}|^3].$$

- Examples
  - This is one example from a simple linear regression $X_j = \alpha + \beta z_j + \epsilon_j$ for $j = 1, 2, \ldots$ where z_j are known numbers not all equal and the ϵ_j are i.i.d with mean zero and variance σ^2.
    $\hat{\beta}_n = \sum_{j=1}^n X_j(z_j - \bar{z}_n)/\sum_{j=1}^n (z_j - \bar{z}_n)^2$
    $= \beta + \sum_{j=1}^n \epsilon_j(z_j - \bar{z}_n)/\sum_{j=1}^n (z_j - \bar{z}_n)^2$.

    Assume

    $$\max_{j\leq n}(z_j - \bar{z}_n)^2/\sum_{j=1}^n (z_j - \bar{z}_n)^2 \to 0.$$

    $\Rightarrow \sqrt{n}\sqrt{\frac{\sum_{j=1}^n (z_j - \bar{z}_n)^2}{n}}(\hat{\beta}_n - \beta) \to_d N(0, \sigma^2)$.

– The example is taken from the randomization test for paired comparison. Let (X_j, Y_j) denote the values of jth pairs with X_j being the result of the treatment and $Z_j = X_j - Y_j$. Conditional on $|Z_j| = z_j$, $Z_j = |Z_j| sgn(Z_j)$ is independent taking values $\pm|Z_j|$ with probability $1/2$, when treatment and control have no difference. Conditional on $z_1, z_2, ...,$ the randomization t-test is the t-statistic $\sqrt{n-1}\bar{Z}_n/s_z$ where s_z^2 is $1/n\sum_{j=1}^n (Z_j - \bar{Z}_n)^2$. When

$$\max_{j \le n} z_j^2 / \sum_{j=1}^n z_j^2 \to 0,$$

this statistic has an asymptotic normal distribution $N(0, 1)$.

Delta Method

Theorem 3.15 (Delta method) For random vector X and X_n in R^k , if there exists two constant a_n and μ such that $a_n(X_n - \mu) \to_d X$ and $a_n \to \infty$, then for any function $g : R^k \mapsto R^l$ such that g has a derivative at μ, denoted by $\nabla g(\mu)$

$$a_n(g(X_n) - g(\mu)) \to_d \nabla g(\mu)X.$$

Proof

By the Skorohod representation, we can construct $\tilde{X}_n$ and $\tilde{X}$ such that $\tilde{X}_n \sim_d X_n$ and $\tilde{X} \sim_d X$ ($\sim_d$ means the same distribution) and $a_n(\tilde{X}_n - \mu) \to_{a.s.} \tilde{X}$.

$\Rightarrow$

$$a_n(g(\tilde{X}_n) - g(\mu)) \to_{a.s.} \nabla g(\mu)\tilde{X}$$

$\Rightarrow$

$$a_n(g(X_n) - g(\mu)) \to_d \nabla g(\mu)X$$

- Examples

  – Let $X_1, X_2, ...$ be i.i.d with fourth moment and $s_n^2 = (1/n)\sum_{i=1}^n (X_i - \bar{X}_n)^2$. Denote m_k as the kth moment of X_1 for $k \leq 4$. Note that $s_n^2 = (1/n)\sum_{i=1}^n X_i^2 - (\sum_{i=1}^n X_i/n)^2$ and

$$\sqrt{n}\left[\begin{pmatrix} \bar{X}_n \\ (1/n)\sum_{i=1}^n X_i^2 \end{pmatrix} - \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}\right]$$

$$\to_d N\left(0, \begin{pmatrix} m_2 - m_1 & m_3 - m_1 m_2 \\ m_3 - m_1 m_2 & m_4 - m_2^2 \end{pmatrix}\right),$$

  the Delta method with $g(x, y) = y - x^2$

  $\Rightarrow \sqrt{n}(s_n^2 - Var(X_1)) \to_d N(0, m_4 - (m_2 - m_1^2)^2)$.

– Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be i.i.d bivariate samples with finite fourth moment. One estimate of the correlation among X and Y is

$$\hat{\rho}_n = \frac{s_{xy}}{\sqrt{s_x^2 s_y^2}},$$

where $s_{xy} = (1/n)\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)$, $s_x^2 = (1/n)\sum_{i=1}^n (X_i - \bar{X}_n)^2$ and $s_y^2 = (1/n)\sum_{i=1}^n (Y_i - \bar{Y}_n)^2$. To derive the large sample distribution of $\hat{\rho}_n$, first obtain the large sample distribution of (s_{xy}, s_x^2, s_y^2) using the Delta method then further apply the Delta method with $g(x, y, z) = x/\sqrt{yz}$.

– The example is taken from the Pearson's Chi-square statistic. Suppose that one subject falls into K categories with probabilities $p_1, ..., p_K$, where $p_1 + ... + p_K = 1$. The Pearson's statistic is defined as

$$\chi^2 = n \sum_{k=1}^{K} (\frac{n_k}{n} - p_k)^2 / p_k,$$

which can be treated as $\sum$(observed count – expected count)^2/expected count. Note $\sqrt{n}(n_1/n - p_1, ..., n_K/n - p_K)$ has an asymptotic multivariate normal distribution. Then we can apply the Delta method to $g(x_1, ..., x_K) = \sum_{i=1}^{K} x_k^2$.

U-statistics

- Definition

Definition 3.6 A *U-statistics* associated with $\tilde{h}(x_1, ..., x_r)$ is defined as

$$U_n = \frac{1}{r!\binom{n}{r}} \sum_{\beta} \tilde{h}(X_{\beta_1}, ..., X_{\beta_r}),$$

where the sum is taken over the set of all unordered subsets β of r different integers chosen from $\{1, ..., n\}$.

- Examples

  – One simple example is $\tilde{h}(x, y) = xy$. Then $U_n = (n(n-1))^{-1}\sum_{i \neq j} X_i X_j$.

  – $U_n = E[\tilde{h}(X_1, ..., X_r)|X_{(1)}, ..., X_{(n)}]$.

  – U_n is the summation of non-independent random variables.

  – If define $h(x_1, ..., x_r)$ as $(r!)^{-1}\sum_{(\tilde{x}_1,...,\tilde{x}_r)} \tilde{h}(\tilde{x}_1, ..., \tilde{x}_r)$, then $h(x_1, ..., x_r)$ is permutation-symmetric

$$U_n = \frac{1}{\binom{n}{r}} \sum_{\beta_1 < ... < \beta_r} h(\beta_1, ..., \beta_r).$$

  – h is called the *kernel* of the U-statistic U_n.

- CLT for U-statistics

Theorem 3.16 Let $\mu = E[h(X_1, ..., X_r)]$. If $E[h(X_1, ..., X_r)^2] < \infty$, then

$$\sqrt{n}(U_n - \mu) - \sqrt{n}\sum_{i=1}^{n} E[U_n - \mu | X_i] \to_p 0.$$

Consequently, $\sqrt{n}(U_n - \mu)$ is asymptotically normal with mean zero and variance $r^2\sigma^2$, where, with $X_1, ..., X_r, \tilde{X}_1, ..., \tilde{X}_r$ i.i.d variables,

$$\sigma^2 = Cov(h(X_1, X_2, ..., X_r), h(X_1, \tilde{X}_2, ..., \tilde{X}_r)).$$

- Some preparation
  - Linear space of r.v.:let $\mathcal{S}$ be a linear space of random variables with finite second moments that contain the constants; i.e., $1 \in \mathcal{S}$ and for any $X, Y \in \mathcal{S}$, $aX + bY \in \mathcal{S}_n$ where a and b are constants.
  - Projection: for random variable T, a random variable S is called the *projection* of T on $\mathcal{S}$ if $E[(T-S)^2]$ minimizes $E[(T-\tilde{S})^2], \tilde{S} \in \mathcal{S}$.

Proposition 3.7 Let $\mathcal{S}$ be a linear space of random variables with finite second moments. Then S is the projection of T on $\mathcal{S}$ if and only if $S \in \mathcal{S}$ and for any $\tilde{S} \in \mathcal{S}$, $E[(T - S)\tilde{S}] = 0$. Every two projections of T onto $\mathcal{S}$ are almost surely equal. If the linear space $\mathcal{S}$ contains the constant variable, then $E[T] = E[S]$ and $Cov(T - S, \tilde{S}) = 0$ for every $\tilde{S} \in \mathcal{S}$.

Proof For any S and $\tilde{S}$ in $\mathcal{S}$,

$$E[(T-\tilde{S})^2] = E[(T-S)^2] + 2E[(T-S)\tilde{S}] + E[(S-\tilde{S})^2].$$

$\Rightarrow$ if S satisfies that $E[(T-S)\tilde{S}] = 0$, then
$E[(T-\tilde{S})^2] \geq E[(T-S)^2]$. $\Rightarrow$ S is the projection of T on $\mathcal{S}$.

If S is the projection, for any constant α, $E[(T-S-\alpha\tilde{S})^2]$ is minimized at $\alpha = 0$. Calculate the derivative at $\alpha = 0$ $\Rightarrow$
$E[(T-S)\tilde{S}] = 0$.

If T has two projections S_1 and S_2, $\Rightarrow$ $E[(S_1-S_2)^2] = 0$. Thus, $S_1 = S_2, a.s.$ If the linear space $\mathcal{S}$ contains the constant variable, choose $\tilde{S} = 1$ $\Rightarrow$ $0 = E[(T-S)\tilde{S}] = E[T] - E[S]$. Clearly,
$Cov(T-S, \tilde{S}) = E[(T-S)\tilde{S}] = 0$.

- Equivalence with projection

Proposition 3.8 Let $\mathcal{S}_n$ be linear space of random variables with finite second moments that contain the constants. Let T_n be random variables with projections S_n on to $\mathcal{S}_n$. If $Var(T_n)/Var(S_n) \to 1$ then

$$Z_n \equiv \frac{T_n - E[T_n]}{\sqrt{Var(T_n)}} - \frac{S_n - E[S_n]}{\sqrt{Var(S_n)}} \to_p 0.$$

Proof . $E[Z_n] = 0$. Note that

$$Var(Z_n) = 2 - 2\frac{Cov(T_n, S_n)}{\sqrt{Var(T_n)Var(S_n)}}.$$

Since S_n is the projection of T_n,
$Cov(T_n, S_n) = Cov(T_n - S_n, S_n) + Var(S_n) = Var(S_n)$. We have

$$Var(Z_n) = 2(1 - \sqrt{\frac{Var(S_n)}{Var(T_n)}}) \to 0.$$

By the Markov's inequality, we conclude that $Z_n \to_p 0$.

- Conclusion
  - if S_n is the summation of i.i.d random variables such that $(S_n - E[S_n])/\sqrt{Var(S_n)} \to_d N(0, \sigma^2)$, so is $(T_n - E[T_n])/\sqrt{Var(T_n)}$. The limit distribution of U-statistics is derived using this lemma.

- Proof of CLT for U-statistics

Proof

Let $\tilde{X}_1, ..., \tilde{X}_r$ be random variables with the same distribution as X_1 and they are independent of $X_1, ..., X_n$. Denote $\tilde{U}_n$ by $\sum_{i=1}^n E[U-\mu|X_i]$.

We show that $\tilde{U}_n$ is the projection of U_n on the linear space $\mathcal{S}_n = \left\{g_1(X_1) + ... + g_n(X_n) : E[g_k(X_k)^2] < \infty, k = 1, ..., n\right\}$, which contains the constant variables. Clearly, $\tilde{U}_n \in \mathcal{S}_n$. For any $g_k(X_k) \in \mathcal{S}_n$,

$$E[(U_n - \tilde{U}_n)g_k(X_k)] = E[E[U_n - \tilde{U}_n|X_k]g_k(X_k)] = 0.$$

$$\tilde{U}_n = \sum_{i=1}^{n} \frac{\binom{n-1}{r-1}}{\binom{n}{r}} E[h(\tilde{X}_1, ..., \tilde{X}_{r-1}, X_i) - \mu | X_i]$$

$$= \frac{r}{n} \sum_{i=1}^{n} E[h(\tilde{X}_1, ..., \tilde{X}_{r-1}, X_i) - \mu | X_i].$$

$\Rightarrow$

$$Var(\tilde{U}_n) = \frac{r^2}{n^2} \sum_{i=1}^{n} E[(E[h(\tilde{X}_1, ..., \tilde{X}_{r-1}, X_i) - \mu | X_i])^2]$$

$$= \frac{r^2}{n} Cov(E[h(\tilde{X}_1, ..., \tilde{X}_{r-1}, X_1)|X_1], E[h(\tilde{X}_1, ..., \tilde{X}_{r-1}, X_1)|X_1])$$

$$= \frac{r^2}{n} Cov(h(X_1, \tilde{X}_2, ..., \tilde{X}_r), h(X_1, X_2 ..., X_r)) = \frac{r^2\sigma^2}{n}.$$

Furthermore,

$$
\begin{aligned}
& Var(U_n) \\
= \;& \binom{n}{r}^{-2} \sum_{\beta} \sum_{\beta'} Cov(h(X_{\beta_1}, ..., X_{\beta_r}), h(X_{\beta'_1}, ..., X_{\beta'_r})) \\
= \;& \binom{n}{r}^{-2} \sum_{k=1}^{r} \sum_{\beta \text{ and } \beta' \text{ share } k \text{ components}} \\
& Cov(h(X_1, X_2, .., X_k, X_{k+1}, ..., X_r), h(X_1, X_2, ..., X_k, \tilde{X}_{k+1}, ..., \tilde{X}_r)).
\end{aligned}
$$

$\Rightarrow Var(U_n) = \sum_{k=1}^{r} \frac{r!}{k!(r-k)!} \frac{(n-r)(n-r+1)\cdots(n-2r+k+1)}{n(n-1)\cdots(n-r+1)} c_k.$

$\Rightarrow Var(U_n) = \frac{r^2}{n} Cov(h(X_1, X_2, ..., X_r), h(X_1, \tilde{X}_2, ..., \tilde{X}_r)) + O(\frac{1}{n^2}).$
$\Rightarrow Var(U_n)/Var(\tilde{U}_n) \to 1.$

$\Rightarrow \frac{U_n - \mu}{\sqrt{Var(U_n)}} - \frac{\tilde{U}_n}{\sqrt{Var(\tilde{U}_n)}} \to_p 0.$

- Example

  - In a bivariate i.i.d sample $(X_1, Y_1), (X_2, Y_2), ...$, one statistic of measuring the agreement is called *Kendall's τ-statistic*

$$\hat{\tau} = \frac{4}{n(n-1)} \sum\sum_{i<j} I\left\{(Y_j - Y_i)(X_j - X_i) > 0\right\} - 1.$$

$\Rightarrow$ $\hat{\tau} + 1$ is a U-statistic of order 2 with the kernel

$$2I\left\{(y_2 - y_1)(x_2 - x_1) > 0\right\}.$$

$\Rightarrow$ $\sqrt{n}(\hat{\tau}_n + 1 - 2P((Y_2 - Y_1)(X_2 - X_1) > 0))$ has an asymptotic normal distribution with mean zero.

Rank Statistics

- Some definitions
  - $X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$ is called *order statistics*
  - The *rank statistics*, denoted by $R_1, ..., R_n$ are the ranks of X_i among $X_1, ..., X_n$. Thus, if all the X's are different, $X_i = X_{(R_i)}$.
  - When there are ties, R_i is defined as the average of all indices such that $X_i = X_{(j)}$ (sometimes called *midrank*).
  - Only consider the case that X's have continuous densities.

- More definitions
  - a *rank statistic* is any function of the ranks
  - a linear rank statistic is a rank statistic of the special form $\sum_{i=1}^{n} a(i, R_i)$ for a given matrix $(a(i, j))_{n \times n}$.
  - if $a(i, j) = c_i a_j$, then such statistic with form $\sum_{i=1}^{n} c_i a_{R_i}$ is called *simple linear rank statistic*: c and a's are called the *coefficients* and *scores*.

- Examples
  - In two independent sample $X_1, ..., X_n$ and $Y_1, ..., Y_m$, a Wilcoxon statistic is defined as the summation of all the ranks of the second sample in the pooled data $X_1, ..., X_n, Y_1, ..., Y_m$, i.e.,

$$W_n = \sum_{i=n+1}^{n+m} R_i.$$

    Other choices for rank statistics: for instance, the van der Waerden statistic $\sum_{i=n+1}^{n+m} \Phi^{-1}(R_i)$.

- Properties of rank statistics

Proposition 3.9 Let $X_1, ..., X_n$ be a random sample from continuous distribution function F with density f. Then

1. the vectors $(X_{(1)}, ..., X_{(n)})$ and $(R_1, ..., R_n)$ are independent;
2. the vector $(X_{(1)}, ..., X_{(n)})$ has density $n! \prod_{i=1}^n f(x_i)$ on the set $x_1 < ... < x_n$;
3. the variable $X_{(i)}$ has density $\binom{n-1}{i-1} F(x)^{i-1}(1-F(x))^{n-i} f(x)$; for F the uniform distribution on $[0,1]$, it has mean $i/(n+1)$ and variance $i(n-i+1)/[(n+1)^2(n+2)]$;

4. the vector $(R_1, ..., R_n)$ is uniformly distributed on the set of all $n!$ permutations of $1, 2, ..., n$;

5. for any statistic T and permutation $r = (r_1, ..., r_n)$ of $1, 2, ..., n$,

$$E[T(X_1, ..., X_n)|(R_1, .., R_n) = r] = E[T(X_{(r_1)}, .., X_{(r_n)})];$$

6. for any simple linear rank statistic $T = \sum_{i=1}^n c_i a_{R_i}$,

$$E[T] = n\bar{c}_n\bar{a}_n, \;\; Var(T) = \frac{1}{n-1}\sum_{i=1}^n (c_i - \bar{c}_n)^2 \sum_{i=1}^n (a_i - \bar{a}_n)^2.$$

- CLT of rank statistics

Theorem 3.17 Let $T_n = \sum_{i=1}^n c_i a_{R_i}$ such that

$$\max_{i\leq n} |a_i - \bar{a}_n| / \sqrt{\sum_{i=1}^n (a_i - \bar{a}_n)^2} \to 0, \quad \max_{i\leq n} |c_i - \bar{c}_n| / \sqrt{\sum_{i=1}^n (c_i - \bar{c}_n)^2} \to 0.$$

Then $(T_n - E[T_n])/\sqrt{Var(T_n)} \to_d N(0,1)$ if and only if for every $\epsilon > 0$,

$$\sum_{(i,j)} I\left\{\sqrt{n}\frac{|a_i - \bar{a}_n||c_i - \bar{c}_n|}{\sqrt{\sum_{i=1}^n (a_i - \bar{a}_n)^2 \sum_{i=1}^n (c_i - \bar{c}_n)^2}} > \epsilon\right\}$$

$$\times \frac{|a_i - \bar{a}_n|^2 |c_i - \bar{c}_n|^2}{\sum_{i=1}^n (a_i - \bar{a}_n)^2 \sum_{i=1}^n (c_i - \bar{c}_n)^2} \to 0.$$

- More on rank statistics
  - a simple linear *signed rank statistic*

$$\sum_{i=1}^{n} a_{R_i^+} \mathrm{sign}(X_i),$$

where $R_1^+, ..., R_n^+$, *absolute rank*, are the ranks of $|X_1|, ..., |X_n|$.

  - In a bivariate sample $(X_1, Y_1), ..., (X_n, Y_n)$, $\sum_{i=1}^n a_{R_i} b_{S_i}$ where $(R_1, ..., R_n)$ and $(S_1, ..., S_n)$ are respective ranks of $(X_1, ..., X_n)$ and $(Y_1, ..., Y_n)$.

Martingales

Definition 3.7 Let $\{Y_n\}$ be a sequence of random variables and $\mathcal{F}_n$ be sequence of σ-fields such that $\mathcal{F}_1 \subset \mathcal{F}_2 \subset ....$ Suppose $E[|Y_n|] < \infty$. Then the pairs $\{(Y_n, \mathcal{F}_n)\}$ is called a *martingale* if

$$E[Y_n|\mathcal{F}_{n-1}] = Y_{n-1}, \quad a.s.$$

$\{(Y_n, \mathcal{F}_n)\}$ is a *submartingale* if

$$E[Y_n|\mathcal{F}_{n-1}] \geq Y_{n-1}, \quad a.s.$$

$\{(Y_n, \mathcal{F}_n)\}$ is a *supmartingale* if

$$E[Y_n|\mathcal{F}_{n-1}] \leq Y_{n-1}, \quad a.s.$$

- Some notes on definition
  - $Y_1, ..., Y_n$ are measurable in $\mathcal{F}_n$. Sometimes, we say Y_n is adapted to $\mathcal{F}_n$.
  - One simple example: $Y_n = X_1 + ... + X_n$, where $X_1, X_2, ...$ are i.i.d with mean zero, and $\mathcal{F}_n$ is the σ-filed generated by $X_1, ..., X_n$.

- Convex function of martingales

Proposition 3.9 Let $\{(Y_n, \mathcal{F}_n)\}$ be a martingale. For any measurable and convex function ϕ, $\{(\phi(Y_n), \mathcal{F}_n)\}$ is a submartingale.

Proof Clearly, $\phi(Y_n)$ is adapted to $\mathcal{F}_n$. It is sufficient to show

$$E[\phi(Y_n)|\mathcal{F}_{n-1}] \geq \phi(Y_{n-1}).$$

This follows from the well-known *Jensen's inequality*: for any convex function ϕ,

$$E[\phi(Y_n)|\mathcal{F}_{n-1}] \geq \phi(E[Y_n|\mathcal{F}_{n-1}]) = \phi(Y_{n-1}).$$

- Jensen's inequality

Proposition 3.10 For any random variable X and any convex measurable function ϕ,

$$E[\phi(X)] \geq \phi(E[X]).$$

Proof

Claim that for any x_0, there exists a constant k_0 such that for any x, $\phi(x) \geq \phi(x_0) + k_0(x - x_0)$.

By the convexity, for any $x' < y' < x_0 < y < x$,

$$\frac{\phi(x_0) - \phi(x')}{x_0 - x'} \leq \frac{\phi(y) - \phi(x_0)}{y - x_0} \leq \frac{\phi(x) - \phi(x_0)}{x - x_0}.$$

Thus, $\frac{\phi(x)-\phi(x_0)}{x-x_0}$ is bounded and decreasing as x decreases to x_0.
Let the limit be $k_0^+ \Rightarrow \frac{\phi(x)-\phi(x_0)}{x-x_0} \geq k_0^+$. $\Rightarrow$
$\phi(x) \geq k_0^+(x - x_0) + \phi(x_0)$.

Similarly,

$$\frac{\phi(x') - \phi(x_0)}{x' - x_0} \le \frac{\phi(y') - \phi(x_0)}{y' - x_0} \le \frac{\phi(x) - \phi(x_0)}{x - x_0}.$$

Then $\frac{\phi(x')-\phi(x_0)}{x'-x_0}$ is increasing and bounded as x' increases to x_0.
Let the limit be $k_0^- \Rightarrow$

$$\phi(x') \ge k_0^-(x' - x_0) + \phi(x_0).$$

Clearly, $k_0^+ \ge k_0^-$. Combining those two inequalities,

$$\phi(x) \ge \phi(x_0) + k_0(x - x_0)$$

for $k_0 = (k_0^+ + k_0^-)/2$.

Choose $x_0 = E[X]$ then $\phi(X) \ge \phi(E[X]) + k_0(X - E[X])$.

- Decomposition of submartingale
  - $Y_n = (Y_n - E[Y_n|\mathcal{F}_{n-1}]) + E[Y_n|\mathcal{F}_{n-1}]$
  - any submartingale can be written as the summation of a martingale and a random variable predictable in $\mathcal{F}_{n-1}$.

- Convergence of martingales

Theorem 3.18 (Martingale Convergence Theorem)
Let $\{(X_n, \mathcal{F}_n)\}$ be submartingale. If
$K = \sup_n E[|X_n|] < \infty$, then $X_n \to_{a.s.} X$ where X is a random variable satisfying $E[|X|] \leq K$.

Corollary 3.1 If $\mathcal{F}_n$ is increasing σ-field and denote $\mathcal{F}_\infty$ as the σ-field generated by $\cup_{n=1}^{\infty}\mathcal{F}_n$, then for any random variable Z with $E[|Z|] < \infty$, it holds

$$E[Z|\mathcal{F}_n] \rightarrow_{a.s.} E[Z|\mathcal{F}_\infty].$$

- CLT for martingale

Theorem 3.19 (Martingale Central Limit Theorem) Let $(Y_{n1}, \mathcal{F}_{n1}), (Y_{n2}, \mathcal{F}_{n2}), ...$ be a martingale. Define $X_{nk} = Y_{nk} - Y_{n,k-1}$ with $Y_{n0} = 0$ thus $Y_{nk} = X_{n1} + ... + X_{nk}$. Suppose that

$$\sum_k E[X_{nk}^2 | \mathcal{F}_{n,k-1}] \to_p \sigma^2$$

where σ is a positive constant and that

$$\sum_k E[X_{nk}^2 I(|X_{nk}| \geq \epsilon) | \mathcal{F}_{n,k-1}] \to_p 0$$

for each $\epsilon > 0$. Then

$$\sum_k X_{nk} \to_d N(0, \sigma^2).$$

Some Notation

- $o_p(1)$ and $O_p(1)$
  - $X_n = o_p(1)$ denotes that X_n converges in probability to zero,
  - $X_n = O_p(1)$ denotes that X_n is bounded in probability; i.e.,

$$\lim_{M\to\infty} \limsup_n P(|X_n| \geq M) = 0.$$

  - for a sequence of random variable $\{r_n\}$, $X_n = o_p(r_n)$ means that $|X_n|/r_n \to_p 0$ and $X_n = O_p(r_n)$ means that $|X_n|/r_n$ is bounded in probability.

- Algebra in $o_p(1)$ and $O_p(1)$

$$o_p(1) + o_p(1) = o_p(1) \quad O_p(1) + O_p(1) = O_p(1),$$
$$O_p(1)o_p(1) = o_p(1) \quad (1 + o_p(1))^{-1} = 1 + o_p(1)$$
$$o_p(R_n) = R_n o_p(1) \quad O_p(R_n) = R_n O_p(1)$$
$$o_p(O_p(1)) = o_p(1).$$

If a real function $R(\cdot)$ satisfies that $R(h) = o(|h|^p)$ as $h \to 0$, $\Rightarrow$ $R(X_n) = o_p(|X_n|^p)$.

If $R(h) = O(|h|^p)$ as $h \to 0$, $\Rightarrow$ $R(X_n) = O_p(|X_n|^p)$.

READING MATERIALS: Lehmann and Casella, Section 1.8, Ferguson, Part 1, Part 2, Part 3 12-15