Density function of $Y = X^2$, where $X \sim N(\mu, 1)$

(A proof of the formula from Lecture Notes, page 10, below Corllary 1.1)

Proposition. Let $X \sim N(\mu,1)$ and $Y = X^2$, $\delta = \mu^2$. Then

$$f_Y(y) = \sum_{k=0}^{\infty} p_k(\delta/2) g(y; (2k+1)/2, 1/2)$$

where $p_k(\delta/2) = \exp(-\delta/2)(\delta/2)^k/k!$ and $g(y;(2k+1)/2,1/2)$ is the density of $Gamma((2k+1)/2, 1/2)$.

Proof. The CDF of Y can be computed as follows. Since Y assumes nonnegative values only, let $y \geq 0$.

$$\begin{aligned} F_Y(y) &= Pr(Y \leq y) = Pr(X^2 \leq y) = Pr(-\sqrt{y} \leq X \leq \sqrt{y}) = Pr(X \leq \sqrt{y}) - Pr(X < -\sqrt{y}) \\ &= Pr(X \leq \sqrt{y}) - Pr(X \leq -\sqrt{y}) \qquad \text{(X cont. random variable)} \\ &= F_X(\sqrt{y}) - F_X(-\sqrt{y}). \end{aligned}$$

Differentiating with respect to y, we get the density function

$$\begin{aligned} f_Y(y) &= \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y}) \\ &= \frac{1}{2\sqrt{y}} \left\{ \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(\sqrt{y}-\mu)^2\right] + \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}(-\sqrt{y}-\mu)^2\right]\right\} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} \exp(-\mu^2/2)\exp(-y/2)\left[\exp(\mu\sqrt{y}) + \exp(-\mu\sqrt{y})\right] \\ & \qquad \text{(by using the power series (Taylor) expansion of the exponential function)} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} \exp(-\mu^2/2)\exp(-y/2) \sum_{n=0,\, n\, even}^{\infty} 2\frac{1}{n!}(\mu\sqrt{y})^n \\ &= \frac{1}{\sqrt{2\pi}} \exp(-\mu^2/2)\exp(-y/2) \sum_{k=0}^{\infty} \frac{1}{(2k)!}\mu^{2k} y^{\frac{2k-1}{2}} \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{\exp(-\mu^2/2)\mu^{2k}}{2\cdot 4\cdot 6\cdot \ldots \cdot (2k-2)(2k)} \left[\frac{1}{1\cdot 3\cdot 5\cdot \ldots \cdot (2k-3)(2k-1)} y^{\frac{2k-1}{2}} \exp(-y/2)\right] \\ &= \sum_{k=0}^{\infty} \frac{\exp(-\mu^2/2)\mu^{2k}}{2^k(k!)} \left[\left(\frac{1}{2}\right)^{k+\frac{1}{2}} \frac{1}{\frac{2k-1}{2}\frac{2k-3}{2}\cdots\frac{3}{2}\frac{1}{2}\Gamma(1/2)} y^{\frac{2k-1}{2}} \exp(-y/2)\right] \\ & \qquad \text{(by using } \Gamma(1/2) = \sqrt{\pi}\text{)} \\ &= \sum_{k=0}^{\infty} \frac{\exp(-\mu^2/2)(\mu^2/2)^k}{(k!)} \left[\frac{1}{\Gamma(\frac{2k+1}{2})}\left(\frac{1}{2}\right)^{\frac{2k+1}{2}} y^{\frac{2k+1}{2}-1} \exp(-y/2)\right] \\ & \qquad \left(\text{by repeated use of } \Gamma(x+1) = x\Gamma(x), \text{ where } x = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{2k-1}{2}\right) \\ &= \sum_{k=0}^{\infty} p_k(\delta/2) g(y; (2k+1)/2, 1/2) \end{aligned}$$

$\square$