Density function of $Y = X^2$, where $X \sim N(\mu, 1)$

*(A proof of the formula from Lecture Notes, page 10, below Corollary 1.1)*

**Proposition.** Let $X \sim N(\mu, 1)$ and $Y = X^2$, $\delta = \mu^2$. Then

$$f_Y(y) = \sum_{k=0}^{\infty} p_k(\delta/2)g(y; (2k + 1)/2, 1/2)$$

where $p_k(\delta/2) = \exp(-\delta/2)(\delta/2)^k/k!$ and $g(y; (2k + 1)/2, 1/2)$ is the density of $\text{Gamma}((2k + 1)/2, 1/2)$.

**Proof.** The CDF of $Y$ can be computed as follows. Since $Y$ assumes nonnegative values only, let $y \geq 0$.

$$F_Y(y) = Pr(Y \leq y) = Pr(X^2 \leq y) = Pr(-\sqrt{y} \leq X \leq \sqrt{y}) = Pr(X \leq \sqrt{y}) - Pr(X < -\sqrt{y})$$

$$= Pr(X \leq \sqrt{y}) - Pr(X \leq -\sqrt{y}) \quad \text{(X cont. random variable)}$$

$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}).$$

Differentiating with respect to $y$, we get the density function

$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}) + \frac{1}{2\sqrt{y}} f_X(-\sqrt{y})$$

$$= \frac{1}{2\sqrt{y}} \left\{ \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2}(\sqrt{y} - \mu)^2 \right] + \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2}(-\sqrt{y} - \mu)^2 \right] \right\}$$

$$= \frac{1}{2\sqrt{2\pi}} \frac{1}{2\sqrt{y}} \exp(-\mu^2/2) \exp(-y/2) \left[ \exp(\mu\sqrt{y}) + \exp(-\mu\sqrt{y}) \right]$$

(by using the power series (Taylor) expansion of the exponential function)

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{2\sqrt{y}} \exp(-\mu^2/2) \exp(-y/2) \sum_{n=0, n \text{ even}}^{\infty} \frac{1}{n!} (\mu\sqrt{y})^n$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \exp(-\mu^2/2) \mu^{2k} \frac{1}{(2k)!} \frac{1}{2 \cdot 4 \cdot 6 \ldots (2k-2)(2k)} \sum_{n=0}^{\infty} \frac{1}{2k+1} \exp(-y/2)$$

(by using $\Gamma(1/2) = \sqrt{\pi}$)

$$= \sum_{k=0}^{\infty} \frac{\exp(-\mu^2/2) \mu^{2k}}{2k!(2k-1)!} \left[ \frac{1}{2} \right]^{k+\frac{1}{2}} \frac{1}{2k-1} \sum_{n=0}^{\infty} \frac{1}{2k+1} \exp(-y/2)$$

(by repeated use of $\Gamma(x+1) = x\Gamma(x)$, where $x = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{2k-1}{2}$)

$$= \sum_{k=0}^{\infty} p_k(\delta/2)g(y; (2k + 1)/2, 1/2)$$