Introduction to Empirical Processes and Semiparametric Inference Lecture 06: Metric Spaces

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Introduction to Part II

The goal of Part II is to provide an in depth coverage of the basics of empirical process techniques which are useful in statistics:

- Chapter 6: mathematical background, metric spaces, outer expectation, linear operators and functional differentiation.
- Chapter 7: stochastic convergence, weak convergence, other modes of convergence.
- Chapter 8: empirical process techniques, maximal inequalities, symmetrization, Glivenk-Canteli results, Donsker results.
- Chapter 9: entropy calculations, VC classes, Glivenk-Canteli and Donsker preservation.
- Chapter 10: empirical process bootstrap.

- Chapter 11: additional empirical process results.
- Chapter 12: the functional delta method.
- Chapter 13: Z-estimators.
- Chapter 14: M-estimators.
- Chapter 15: Case-studies II.

Topological Spaces

A collection $\mathcal{O}$ of subsets of a set X is a *topology in* X if:

(i) $\emptyset \in \mathcal{O}$ and $X \in \mathcal{O}$, where $\emptyset$ is the empty set;

(ii) If $U_j \in \mathcal{O}$ for $j = 1, \ldots, m$, then $\bigcap_{j=1,\ldots,m} U_j \in \mathcal{O}$;

(iii) If $\{U_\alpha\}$ is an arbitrary collection of members of $\mathcal{O}$ (finite, countable or uncountable), then $\bigcup_\alpha U_\alpha \in \mathcal{O}$.

When $\mathcal{O}$ is a topology in X, then X (or the pair $(X, \mathcal{O})$) is a *topological space*, and the members of $\mathcal{O}$ are called the *open sets* in X.

For a subset $A \subset X$, the *relative topology* on A consists of the sets $\{A \cap B : B \in \mathcal{O}\}$: check that this is a topology.

A map $f : X \mapsto Y$ between topological spaces is *continuous* if $f^{-1}(U)$ is open in X whenever U is open in Y.

A set B in X is *closed* if and only if its complement in X, denoted $X - B$, is open.

The *closure* of an arbitrary set $E \in X$, denoted $\overline{E}$, is the smallest closed set containing E.

The *interior* of an arbitrary set $E \in X$, denoted E°, is the largest open set contained in E.

A subset A of a topological space X is *dense* if $\overline{A} = X$.

A topological space X is *separable* if it has a countable dense subset.

A *neighborhood* of a point $x \in X$ is any open set that contains x.

A topological space is *Hausdorff* if distinct points in X have disjoint neighborhoods.

A sequence of points $\{x_n\}$ in a topological space X *converges* to a point $x \in X$, denoted $x_n \to x$, if every neighborhood of x contains all but finitely many of the x_n.

Suppose $x_n \to x$ and $x_n \to y$.

Then x and y share all neighborhoods, and $x = y$ when X is Hausdorff.

If a map $f : X \mapsto Y$ between topological spaces is continuous, then $f(x_n) \to f(x)$ whenever $x_n \to x$ in X; to see this,

- Let $\{x_n\} \subset X$ be a sequence with $x_n \to x \in X$.
- Then for any open $U \subset Y$ containing $f(x)$, all but finitely many $\{x_n\}$ are in $f^{-1}(U)$, and thus all but finitely many $\{f(x_n)\}$ are in U.
- Since U was arbitrary, we have $f(x_n) \to f(x)$.

We now review the important concept of *compactness*:

- A subset K of a topological space is *compact* if for every set $A \supset K$, where A is the union of a collection of open sets $\mathcal{S}$, K is also contained in some finite union of sets in $\mathcal{S}$.
- When the topological space involved is also Hausdorff, then compactness of K is equivalent to the assertion that every sequence in K has a convergent subsequence (converging to a point in K).
- This result implies that compact subsets of Hausdorff topological spaces are necessarily closed.
- Note that a compact set is sometimes called a *compact* for short.
- A σ*-compact* set is a countable union of compacts.

A collection $\mathcal{A}$ of subsets of a set X is a σ*-field in* X (sometimes called a σ*-algebra*) if:

(i) $X \in \mathcal{A}$;

(ii) If $U \in \mathcal{A}$, then $X - U \in \mathcal{A}$;

(iii) The countable union $\bigcup_{j=1}^{\infty} U_j \in \mathcal{A}$ whenever $U_j \in \mathcal{A}$ for all $j \geq 1$.

Note that (iii) clearly includes finite unions.

When (iii) is only required to hold for finite unions, then $\mathcal{A}$ is called a *field*.

When $\mathcal{A}$ is a σ-field in X, then X (or the pair $(X, \mathcal{A})$) is a *measurable space*, and the members of $\mathcal{A}$ are called the *measurable sets* in X.

If X is a measurable space and Y is a topological space, then a map $f : X \mapsto Y$ is *measurable* if $f^{-1}(U)$ is measurable in X whenever U is open in Y.

If $\mathcal{O}$ is a collection of subsets of X (not necessary open), then there exists a smallest σ-field $\mathcal{A}^*$ in X so that $\mathcal{O} \subset \mathcal{A}^*$.

This $\mathcal{A}^*$ is called the σ-field *generated* by $\mathcal{O}$.

To see that such an $\mathcal{A}^*$ exists:

- Let $\mathcal{S}$ be the collection of all σ-fields in X which contain $\mathcal{O}$.
- Since the collection of all subsets of X is one such σ-field, $\mathcal{S}$ is not empty.
- Define $\mathcal{A}^*$ to be the intersection of all $\mathcal{A} \in \mathcal{S}$.
- Clearly, $\mathcal{O} \in \mathcal{A}^*$ and $\mathcal{A}^*$ is in every σ-field containing $\mathcal{O}$.
- All that remains is to show that $\mathcal{A}^*$ is itself a σ-field.

To show this,

- Assume that $A_j \in \mathcal{A}^*$ for all integers $j \geq 1$.
- If $\mathcal{A} \in \mathcal{S}$, then $\bigcup_{j\geq 1} A_j \in \mathcal{A}$.
- Since $\bigcup_{j\geq 1} A_j \in \mathcal{A}$ for every $\mathcal{A} \in \mathcal{S}$, we have $\bigcup_{j\geq 1} A_j \in \mathcal{A}^*$.
- Also $X \in \mathcal{A}^*$ since $X \in \mathcal{A}$ for all $\mathcal{A} \in \mathcal{S}$; and for any $A \in \mathcal{A}^*$, both A and $X - A$ are in every $\mathcal{A} \in \mathcal{S}$.

Thus $\mathcal{A}^*$ is indeed a σ-field.

A σ-field is *separable* if it is generated by a countable collection of subsets.

Note: we have already defined “separable” as a characteristic of certain topological spaces.

There is a connection between the two definitions which we will point out shortly in our discussion on metric spaces.

When X is a topological space, the smallest σ-field $\mathcal{B}$ generated by the open sets is called the *Borel σ-field* of X.

Elements of $\mathcal{B}$ are called *Borel sets*.

A function $f : X \mapsto Y$ between topological spaces is *Borel-measurable* if it is measurable with respect to the Borel σ-field of X.

Clearly, a continuous function between topological spaces is also Borel-measurable.

For a σ-field $\mathcal{A}$ in a set X, a map $\mu : \mathcal{A} \mapsto \bar{\mathbb{R}}$ is a *measure* if:

(i) $\mu(A) \in [0, \infty]$ for all $A \in \mathcal{A}$;

(ii) $\mu(\emptyset) = 0$;

(iii) For any disjoint sequence $\{A_j\} \in \mathcal{A}$,
$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(A_j)$ (*countable additivity*).

The triple $(X, \mathcal{A}, \mu)$ is called a *measure space*.

If $X = A_1 \cup A_2 \cup \cdots$ for some finite or countable sequence of sets in $\mathcal{A}$ with $\mu(A_j) < \infty$ for all indices j, then μ is σ*-finite*.

If $\mu(X) = 1$, then μ is a *probability measure*.

For a probability measure P on a set Ω with σ-field $\mathcal{A}$, the triple $(\Omega, \mathcal{A}, P)$ is called a *probability space*.

If the set $[0, \infty]$ in Part (i) is extended to $(-\infty, \infty]$ or replaced by $[-\infty, \infty)$ (but not both), then μ is a *signed measure*.

For a measure space $(X, \mathcal{A}, \mu)$, let $\mathcal{A}^*$ be the collection of all $E \subset X$ for which there exists $A, B \in \mathcal{A}$ with $A \subset E \subset B$ and $\mu(B - A) = 0$, and define $\mu(E) = \mu(A)$ in this setting.

Then $\mathcal{A}^*$ is a σ-field, μ is still a measure, and $\mathcal{A}^*$ is called the μ-*completion* of $\mathcal{A}$.

Metric Spaces

A *metric space* is a set $\mathbb{D}$ together with a *metric*.

A metric or *distance function* is a map $d : \mathbb{D} \times \mathbb{D} \mapsto [0, \infty)$ where:

(i) $d(x, y) = d(y, x)$;

(ii) $d(x, z) \leq d(x, y) + d(y, z)$ (the *triangle inequality*);

(iii) $d(x, y) = 0$ if and only if $x = y$.

A *semimetric* or *pseudometric* satisfies (i) and (ii) but not necessarily (iii).

Technically, a metric space consists of the pair $(\mathbb{D}, d)$, but usually only $\mathbb{D}$ is given and the underlying metric d is implied by the context.

A semimetric space is also a topological space with the open sets generated by applying arbitrary unions to the *open* r*-balls* $B_r(x) \equiv \{y : d(x, y) < r\}$ for $r \geq 0$ and $x \in \mathbb{D}$ (where $B_0(x) \equiv \emptyset$).

A metric space is also Hausdorff, and, in this case, a sequence $\{x_n\} \in \mathbb{D}$ converges to $x \in \mathbb{D}$ if $d(x_n, x) \to 0$.

For a semimetric space, $d(x_n, x) \to 0$ ensures only that x_n converges to elements in the *equivalence class* of x, where the equivalence class of x consists of all $\{y \in \mathbb{D} : d(x, y) = 0\}$.

Accordingly, the closure $\overline{A}$ of a set $A \in \mathbb{D}$ is not only the smallest closed set containing A, as stated earlier, but $\overline{A}$ also equals the set of all points that are limits of sequences $\{x_n\} \in A$.

In addition, two semimetrics d_1 and d_2 on a set $\mathbb{D}$ are considered equivalent (in a topological sense) if they both generate the same open sets, and equivalent metrics yield the same convergent subsequences.

A map $f : \mathbb{D} \mapsto \mathbb{E}$ between two semimetric spaces is *continuous at a point* x if and only if $f(x_n) \to f(x)$ for every sequence $x_n \to x$.

The map f is continuous (in the topological sense) if and only if it is continuous at all points $x \in \mathbb{D}$.

The following lemma helps to define *semicontinuity* for real valued maps:

LEMMA 1. *Let $f : \mathbb{D} \mapsto \mathbb{R}$ be a function on the metric space $\mathbb{D}$; Then TFAE:*

(i) For all $c \in \mathbb{R}$, the set $\{y : f(y) \geq c\}$ is closed.

(ii) For all $y_0 \in \mathbb{D}$, $\limsup_{y \to y_0} f(y) \leq f(y_0)$.

A function $f : \mathbb{D} \mapsto \mathbb{R}$ satisfying either (i) or (ii) (and hence both) of the conditions in Lemma 1 is said to be *upper semicontinuous*.

A function $f : \mathbb{D} \mapsto \mathbb{R}$ is *lower semicontinuous* if $-f$ is upper semicontinuous.

Using Condition (ii), it is easy to see that a function which is both upper and lower semicontinuous is also continuous.

The set of all continuous and bounded functions $f : \mathbb{D} \mapsto \mathbb{R}$, which we denote $C_b(\mathbb{D})$, plays an important role in weak convergence on the metric space $\mathbb{D}$, which we will explore in Chapter 7.

It is not hard to see that the Borel σ-field on a metric space $\mathbb{D}$ is the smallest σ-field generated by the open balls.

It turns out that the Borel σ-field $\mathcal{B}$ of $\mathbb{D}$ is also the smallest σ-field $\mathcal{A}$ making all of $C_b(\mathbb{D})$ measurable; To see this,

- Note that any closed $A \subset \mathbb{D}$ is the preimage of the closed set $\{0\}$ for the continuous bounded function $x \mapsto d(x, A) \wedge 1$, where for any set $B \subset \mathbb{D}$, $d(x, B) \equiv \inf\{d(x, y) : y \in B\}$.
- Thus $\mathcal{B} \subset \mathcal{A}$.
- Since it is obvious that $\mathcal{A} \subset \mathcal{B}$, we now have $\mathcal{A} = \mathcal{B}$.$\square$

A Borel-measurable map $X : \Omega \mapsto \mathbb{D}$ defined on a probability space $(\Omega, \mathcal{A}, P)$ is called a *random element* or *random map* (or *random variable*) with values in $\mathbb{D}$.

Borel measurability is, in many ways, the natural concept to use on metric spaces since it connects nicely with the topological structure.

A *Cauchy sequence* is a sequence $\{x_n\}$ in a semimetric space $(\mathbb{D}, d)$ such that $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

A semimetric space $\mathbb{D}$ is *complete* if every Cauchy sequence has a limit $x \in \mathbb{D}$.

Every metric space $\mathbb{D}$ has a completion $\overline{\mathbb{D}}$ which has a dense subset *isometric* with $\mathbb{D}$.

Two metric spaces are isometric if there exists a *bijection* (a one-to-one and onto map) between them which preserves distances.

When a metric space $\mathbb{D}$ is separable, and therefore has a countable dense subset, the Borel σ-field for $\mathbb{D}$ is itself a separable σ-field.

To see this, let $A \in \mathbb{D}$ be a countable dense subset and consider the collection of open balls with centers at points in A and with rational radii.

Clearly, the set of such balls is countable and generates all open sets in $\mathbb{D}$:

- Let $x_0 \in \mathbb{D}$, $\{x_n\} \in A$: $x_n \to x_0$, and $\epsilon_n = |x_n - x_0|$.
- It is not hard to see that $\cup_{n \geq 1} \{x : |x - x_n| < (\eta - \epsilon_n) \vee 0\} = B_\eta(x_0)$ for any $\eta > 0$.

A topological space X is *Polish* if it is separable and if there exists a metric making X into a complete metric space.

Hence any complete and separable metric space is Polish.

Furthermore, any open subset of a Polish space is also Polish.

Examples of Polish spaces include Euclidean spaces and many other interesting spaces that we will explore shortly.

A subset K is *totally bounded* if and only if for every $r > 0$, K can be covered by finitely many open r-balls.

Furthermore, a subset K of a complete semimetric space is compact if and only if it is totally bounded and closed.

A totally bounded subset K is also called *precompact* because every sequence in K has a Cauchy subsequence.

This relationship between compactness and total boundedness implies that a σ-compact set in a metric space is separable.

These definitions of compactness agree with the previously given compactness properties for Hausdorff spaces.

This happens because a semimetric space $\mathbb{D}$ can be made into a metric—and hence Hausdorff—space $\mathbb{D}_H$ by equating points in $\mathbb{D}_H$ with equivalence classes in $\mathbb{D}$.

Banach Spaces

A very important example of a metric space is a *normed space*.

A normed space $\mathbb{D}$ is a vector space (also called a linear space) equipped with a *norm*, and a norm is a map $\|\cdot\| : \mathbb{D} \mapsto [0, \infty)$ such that, for all $x, y \in \mathbb{D}$ and $\alpha \in \mathbb{R}$,

(i) $\|x + y\| \leq \|x\| + \|y\|$ (another triangle inequality);

(ii) $\|\alpha x\| = |\alpha| \times \|x\|$;

(iii) $\|x\| = 0$ if and only if $x = 0$.

A *seminorm* satisfies (i) and (ii) but not necessarily (iii).

A normed space is a metric space (and a seminormed space is a semimetric space) with $d(x, y) = \|x - y\|$, for all $x, y \in \mathbb{D}$.

A complete normed space is called a *Banach space*.

Two seminorms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a set $\mathbb{D}$ are equivalent if the following is true for all $x, \{x_n\} \in \mathbb{D}$:

$$\|x_n - x\|_1 \to 0 \text{ if and only if } \|x_n - x\|_2 \to 0.$$

In our definition of a normed space $\mathbb{D}$, we require the space to also be a vector space (and therefore it contains all linear combinations of elements in $\mathbb{D}$).

However, it is sometimes of interest to apply norms to subsets $K \subset \mathbb{D}$ which may not be linear subspaces.

In this setting, let lin K denote the *linear span of* K (all linear combinations of elements in K), and let $\overline{\text{lin}}\, K$ the closure of lin K.

Note that both lin$K$ and $\overline{\text{lin}}K$ are now vector spaces and that $\overline{\text{lin}}K$ is also a Banach space.

We now present several specific examples of metric spaces.//[0.5ex]

The Euclidean space $\mathbb{R}^d$ is a Banach space with squared norm $\|x\|^2 = \sum_{j=1}^d x_j^2$.

This space is equivalent under several other norms, including $\|x\| = \max_{1 \le j \le d} |x_j|$ and $\|x\| = \sum_{j=1}^d |x_j|$.

A Euclidean space is separable with a countably dense subset consisting of all vectors with rational coordinates.

By the Heine-Borel theorem, a subset in a Euclidean space is compact if and only if it is closed and bounded.

The Borel σ-field is generated by the intervals of the type $(-\infty, x]$, for rational x, where the interval is defined as follows: $y \in (-\infty, x]$ if and only if $y_j \in (-\infty, x_j]$ for all coordinates $j = 1, \ldots, d$.

For one-dimensional Euclidean space, $\mathbb{R}$, the norm is $\|x\| = |x|$ (absolute value).

The extended real line $\bar{\mathbb{R}} = [-\infty, \infty]$ is a metric space with respect to the metric $d(x, y) = |G(x) - G(y)|$, where $G : \bar{\mathbb{R}} \mapsto \mathbb{R}$ is any strictly monotone increasing, continuous and bounded function, such as the $\arctan$ function.

For any sequence $\{x_n\} \in \bar{\mathbb{R}}$, $|x_n - x| \to 0$ implies $d(x_n, x) \to 0$, while divergence of $d(x_n, x)$ implies divergence of $|x_n - x|$.

In addition, it is possible for a sequence to converge, with respect to d, to either $-\infty$ or ∞.

This makes $\bar{\mathbb{R}}$ compact.

Another important example is the set of bounded real functions $f : T \mapsto \mathbb{R}$, where T is an arbitrary set.

This is a vector space if sums $z_1 + z_2$ and products with scalars, αz, are defined pointwise for all $z, z_1, z_2 \in \ell^\infty(T)$.

Specifically, $(z_1 + z_2)(t) = z_1(t) + z_2(t)$ and $(\alpha z)(t) = \alpha z(t)$, for all $t \in T$.

This space is denoted $\ell^\infty(T)$.

The *uniform norm* $\|x\|_T \equiv \sup_{t \in T} |x(t)|$ makes $\ell^\infty(T)$ into a Banach space consisting exactly of all functions $z : T \mapsto \mathbb{R}$ satisfying $\|z\|_T < \infty$.

Note that $\ell^\infty(T)$ is separable if and only if T is countable.

Two useful subspaces of $\ell^\infty([a, b])$, where $a, b \in \bar{\mathbb{R}}$, are $C[a, b]$ and $D[a, b]$.

The space $C[a,b]$ consists of continuous functions $z : [a,b] \mapsto \mathbb{R}$, and $D[a,b]$ is the space of *cadlag* functions which are right-continuous with left-hand limits (cadlag is an abbreviation for continue à droite, limites à gauche).

We usually equip these spaces with the uniform norm $\|\cdot\|_{[a,b]}$ inherited from $\ell^\infty([a,b])$.

Note that $C[a,b] \subset D[a,b] \subset \ell^\infty([a,b])$.

Relative to the uniform norm, $C[a, b]$ is separable, and thus also Polish by the completeness established in Exercise 6.5.5(a), but $D[a, b]$ is not separable.

Sometimes, $D[a, b]$ is called the *Skorohod space*, although Skorohod equipped $D[a, b]$ with a special metric—quite different than the uniform metric—resulting in a separable space.

Arzelà-Ascola Theorem

An important subspace of $\ell^\infty(T)$ is the space $UC(T,\rho)$, where ρ is a semimetric on T.

$UC(T,\rho)$ consists of all bounded function $f : T \mapsto \mathbb{R}$ which are uniformly ρ-continuous, i.e.,

$$\lim_{\delta\downarrow 0} \sup_{\rho(s,t)<\delta} |f(s) - f(t)| = 0.$$

When (T,ρ) is totally bounded, the boundedness requirement for functions in $UC(T,\rho)$ is superfluous since a uniformly continuous function on a totally bounded set must necessarily be bounded.

We denote $C(T, \rho)$ to be the space of ρ-continuous (not necessarily continuous) function on T.

The spaces $C[a, b]$, $D[a, b]$, $UC(T, \rho)$, $C(\overline{T}, \rho)$, when (T, ρ) is a totally bounded semimetric space, and $UC(T, \rho)$ and $\ell^\infty(T)$, for an arbitrary set T, are all complete with respect to the uniform metric.

When (T, ρ) is a compact semimetric space, T is totally bounded, and a ρ-continuous function in T is automatically uniformly ρ-continuous.

Thus, when T is compact, $C(T, \rho) = UC(T, \rho)$.

Actually, every space $UC(T, \rho)$ is equivalent to a space $C(\overline{T}, \rho)$, because the completion $\overline{T}$ of a totally bounded space T is compact and, furthermore, every uniformly continuous function on T has a unique continuous extension to $\overline{T}$.

The foregoing structure makes it clear that $UC(T, \rho)$ is a Polish space that is made complete by the uniform norm (and hence is also separable).

Moreover, all compact sets in $\ell^\infty(T)$ have a specific form:

THEOREM 1. *(Arzelà-Ascoli)*

(a) The closure of $K \subset \ell^\infty(T)$ is compact if and only if

(i) $\sup_{x\in K} |x(t_0)| < \infty$, for some $t_0 \in T$; and

(ii) for some semimetric ρ making T totally bounded,

$$\lim_{\delta\downarrow 0} \sup_{x\in K} \sup_{s,t\in T:\rho(s,t)<\delta} |x(s) - x(t)| = 0.$$

(b) The set $K \subset \ell^\infty(T)$ is σ-compact if and only if $K \subset LC(T, \rho)$ for some semimetric ρ making T totally bounded, where $LC(T, \rho)$ is the subset of $UC(T, \rho)$ consisting of all bounded functions x with

$$\sup_{\delta>0} \sup_{s,t\in T:\rho(s,t)<\delta} \frac{|x(s) - x(t)|}{\delta} < \infty.$$

(c) The closure of $K \subset \ell^\infty(T)$ *is separable if and only if* $K \subset UC(T, \rho)$ *for some semimetric* ρ *making* T *totally bounded.*

Since all compact sets are trivially σ-compact, Theorem 1 implies that any compact set in $\ell^\infty(T)$ is actually contained in $LC(T, \rho)$ for some semimetric ρ making T totally bounded.

Another important class of metric spaces are product spaces.

For a pair of metric spaces $(\mathbb{D}, d)$ and $(\mathbb{E}, e)$, the *Cartesian product* $\mathbb{D} \times \mathbb{E}$ is a metric space with respect to the metric

$$\rho((x_1, y_1), (x_2, y_2)) \equiv d(x_1, x_2) \vee e(y_1, y_2),$$

for $x_1, x_2 \in \mathbb{D}$ and $y_1, y_2 \in \mathbb{E}$.

The resulting topology is the *product topology*.

In this setting, convergence of $(x_n, y_n) \to (x, y)$ is equivalent to convergence of both $x_n \to x$ and $y_n \to y$.

There are two natural σ-fields for $\mathbb{D} \times \mathbb{E}$ that we can consider.

The first is the Borel σ-field for $\mathbb{D} \times \mathbb{E}$ generated from the product topology.

The second is the product σ-field generated by all sets of the form $A \times B$, where $A \in \mathcal{A}$, $B \in \mathcal{B}$, and $\mathcal{A}$ and $\mathcal{B}$ are the respective σ-fields for $\mathbb{D}$ and $\mathbb{E}$.

These two are equal when $\mathbb{D}$ and $\mathbb{E}$ are separable, but they may be unequal otherwise, with the first σ-field larger than the second.

Suppose $X : \Omega \mapsto \mathbb{D}$ and $Y : \Omega \mapsto \mathbb{E}$ are Borel-measurable maps defined on a measurable space $(\Omega, \mathcal{A})$.

Then $(X, Y) : \Omega \mapsto \mathbb{D} \times \mathbb{E}$ is a measurable map for the product of the two σ-fields by the definition of a measurable map.

Unfortunately, when the Borel σ-field for $\mathbb{D} \times \mathbb{E}$ is larger than the product σ-field, then it is possible for (X, Y) to not be Borel-measurable.